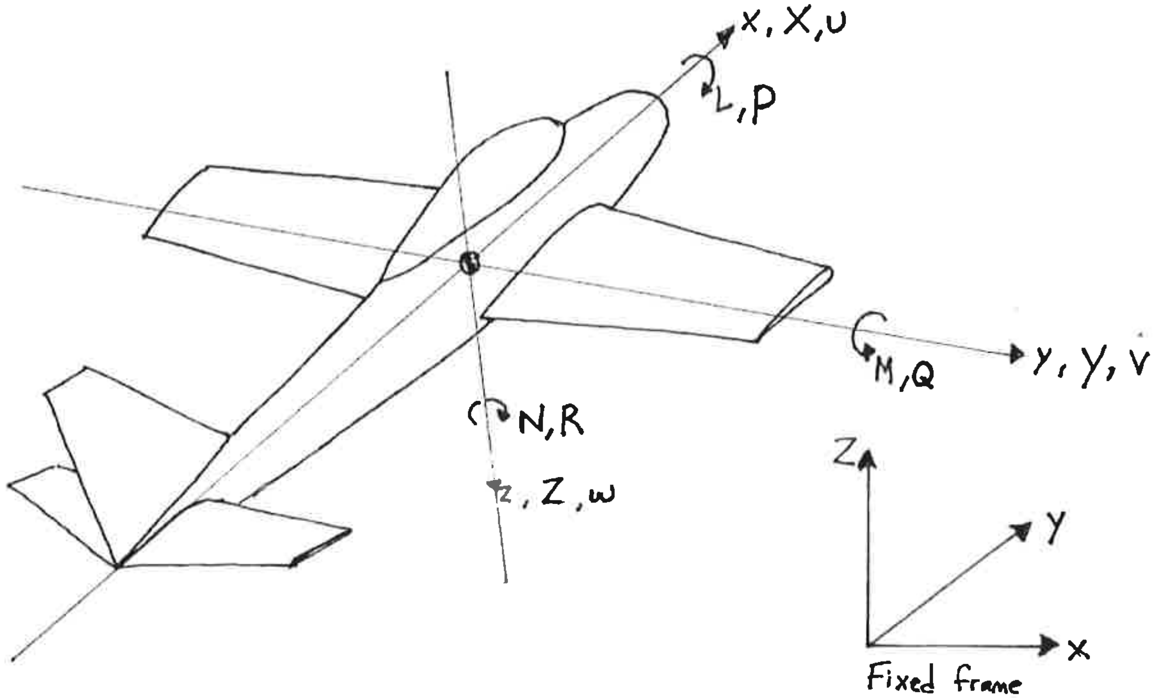
AEM 617

Flight Control Systems

Lecture 9

Aircraft Coordinate System



x, y, z	Aircraft "stability" frame location
X, Y, Z	Forces in stability frame
u, v, w	Velocity in stability frame
L, M, N	moment in stability frame
P, Q, R	Angular velocities (roll, pitch, yaw)
ϕ, θ, ψ	Euler angles (orientation)

Warning:

Outside of aerodynamics, the x axis is usually down the length of the a/c.

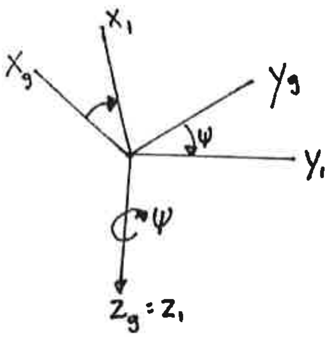
$$X_{loft} = -X_{aero\ stability}$$
$$Y_{loft} = Y_{aero\ stability}$$
$$Z_{loft} = -Z_{aero\ stability}$$

Euler Angles Φ, θ, ψ (order dependent!)

local to global: yaw (ψ), pitch (θ), roll (ϕ)

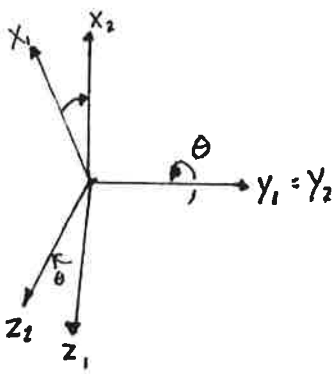
global to local: roll (ϕ), pitch (θ), yaw (ψ)

Yaw



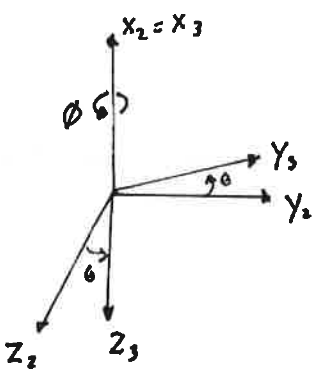
$$\left(\bar{X}_1\right) = \underbrace{\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{T_\psi^{lg}} \left(\bar{X}_G\right)$$

pitch



$$\bar{X}_2 = \underbrace{\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}}_{T_\theta^{lg}} \bar{X}_1$$

Roll



$$\bar{X}_3 = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}}_{T_\phi^{lg}} \bar{X}_2$$

Global

$$\bar{X}_G = \underbrace{T_\psi^{lg}\, T_\theta^{lg}\, T_\phi^{lg}}_{T^{lg}} \bar{X}_L$$

$$T^{lg} = \begin{bmatrix} C_\theta C_\psi & S_\phi S_\theta C_\psi - C_\phi S_\psi & C_\phi S_\theta C_\psi + S_\phi S_\psi \\ C_\theta S_\psi & S_\phi S_\theta S_\psi + C_\phi C_\psi & C_\phi S_\theta S_\psi - S_\phi C_\psi \\ -S_\theta & S_\phi C_\theta & C_\phi C_\theta \end{bmatrix}$$

A beautiful property of transformation matrices is

$$T^{gl} = \left(T^{lg}\right)^{-1} = T^{lg^T}$$

The inverse is the transpose!

Ex:

Given a global frame velocity of $\bar{V} = (1, 0, 0)$ and the orientation Euler angles of $(\phi, \theta, \psi) = (0, 10°, 10°)$, what is the body frame velocity vector (u, v, w)?

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = T^{gl} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = T^{lg^T} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} C_\theta C_\psi \\ S_\phi S_\theta C_\psi - C_\phi S_\psi \\ C_\phi S_\theta C_\psi + S_\phi S_\psi \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0.9698 \\ -0.1413 \\ 0.1986 \end{pmatrix}$$

A demonstration of Gimbal Lock.

The transformation matrix when pitch $\theta = 90^\circ$ is

$$T^{'g} = \begin{bmatrix} 0 & S_\phi C_\psi - C_\phi S_\psi & C_\phi C_\psi + S_\phi S_\psi \\ 0 & S_\phi S_\psi + C_\phi C_\psi & C_\phi S_\psi - S_\phi C_\psi \\ -1 & 0 & 0 \end{bmatrix}$$

Remember that

$$\cos(A \mp B) = \cos(A)\cos(B) \pm \sin(A)\sin(B)$$

and

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$

substitute

$$T^{'g} = \begin{bmatrix} 0 & S(\phi - \psi) & C(\phi - \psi) \\ 0 & C(\phi - \psi) & -S(\phi - \psi) \\ -1 & 0 & 0 \end{bmatrix}$$

Uh-oh! $T^{'g}$ is only ~~a~~ a function of $\phi - \psi$

We can not distinguish between ϕ and ψ when $\theta = 90^\circ$!

Flight Control Systems use mechanical, electrical and hydraulic physics

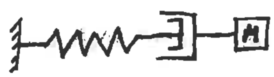
Table 1.2 Ideal System Elements (Linear)

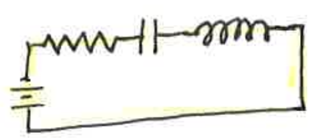
System Type	Mechanical	Electrical	Fluid	Thermal
A-type element	Mass	Capacitor	Fluid capacitor	Thermal capacitor
Elemental equation	$F = m\dfrac{dv}{dt}$	$i = C\dfrac{de}{dt}$	$Q_f = C_f\dfrac{dP}{dt}$	$Q_h = C_h\dfrac{dT}{dt}$
Energy stored	Kinetic	Electric field	Potential	Thermal
Energy equation	$\mathscr{E}_K = \dfrac{m}{2}v^2$	$\mathscr{E}_E = \dfrac{C}{2}e^2$	$\mathscr{E}_P = \dfrac{C_f}{2}P^2$	$\mathscr{E}_T = \dfrac{C_h}{2}T^2$
T-type element	Spring	Inductor	Inertor	None
Elemental equation	$v = \dfrac{1}{k}\dfrac{dF}{dt}$	$e = L\dfrac{di}{dt}$	$P = I\dfrac{dQ_f}{dt}$	
Energy stored	Potential	Magnetic field	Kinetic	
Energy equation	$\mathscr{E}_P = \dfrac{1}{2k}F^2$	$\mathscr{E}_M = \dfrac{L}{2}i^2$	$\mathscr{E}_K = \dfrac{I}{2}Q_f^2$	
D-type element	Damper	Resistor	Fluid resistor	Thermal resistor
Elemental equation	$F = bv$	$i = \left(\dfrac{1}{R}\right)e$	$Q_f = \left(\dfrac{1}{R_f}\right)P$	$Q_h = \left(\dfrac{1}{R_h}\right)T$
	$v = \dfrac{1}{b}F$	$e = Ri$	$P = R_fQ_f$	$T = R_hQ_h$
Energy dissipation rate	$\dfrac{d\mathscr{E}_D}{dt} = Fv$	$\dfrac{d\mathscr{E}_D}{dt} = ie$	$\dfrac{d\mathscr{E}_D}{dt} = Q_fP$	$\dfrac{d\mathscr{E}_D}{dt} = Q_hT$

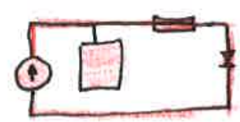
Note: Each A-type variable represents a spatial difference across the element.

Not common for FCS! but can contribute to physics.

Dynamic modeling and Control of Engineering Systems. Shearer, et al.

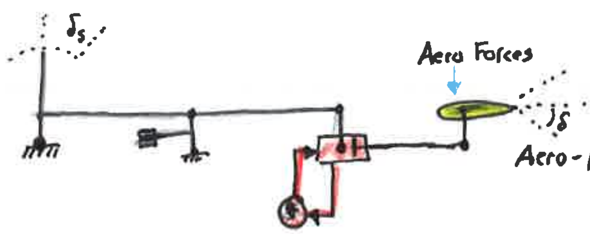






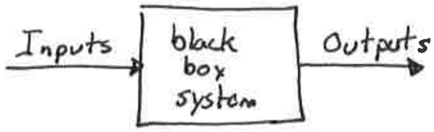
We will need to describe systems where a combination of these physics occurs.

In other words, how can we make a system model of a FCS?

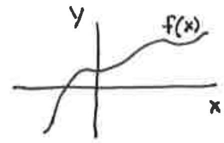


Aero-Mechanical - Hydraulic System

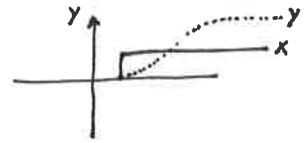
Systems



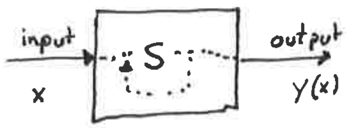
Ex: $y = f(x)$
zero order



Ex: $\dot{y} = f(x)$, $\dot{y} = \frac{dy}{dt}$
1st order



States are the minimum number of variables necessary to describe the system.

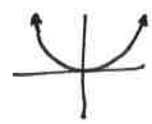


Ex: $y = f(x)$ requires zero states, since given x, y is always $f(x)$

Ex: $\dot{y} = f(x)$ requires 1 state. We must track $y(t)$ since only $\dot{y} = f(x)$

All real systems are in some way non-linear systems.

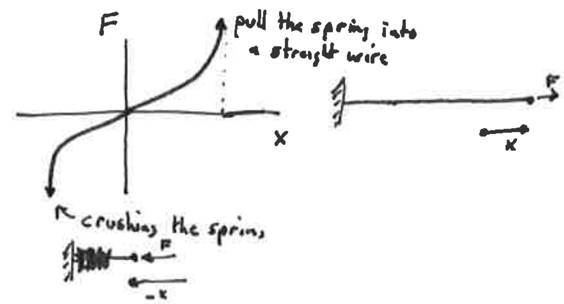
Ex: $y = f(x) = x^2$



Can you think of any system where the output is <u>always</u> a linear system of input?

Ex:

Spring

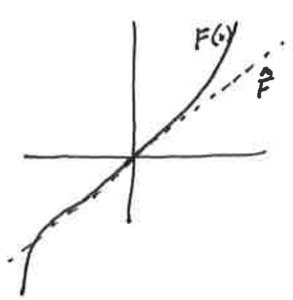


But about the relaxed spring position, the spring behaves in a linear fashion such that $F = Kx$

Expand into a Taylor Series

$$F(\underset{\substack{\uparrow \\ x_0+\Delta x}}{\hat{x}}) = F(x_0) + \left.\frac{dF(x_0)}{dx}\right|_{x_0} \cdot \Delta x + \left.\frac{d^2}{dx^2}F(x)\right|_{x_0} \cdot \Delta x^2$$

So operating about x_0, $\hat{F}(\Delta x) \approx \left.\frac{dF}{dx}\right|_{x_0} \Delta x$ Linearized



State Space Model

States Vector $\begin{pmatrix} x \end{pmatrix}$

System dynamics $\underbrace{\begin{pmatrix} \dot{x} \end{pmatrix}}_{\text{rate of Change in states wrt time}} = \underbrace{F(x)}_{\text{function of states}} + \underbrace{G(u)}_{\text{Input}}$

Output $\underbrace{\begin{pmatrix} y \end{pmatrix}}_{\text{output}} = \underbrace{H(x)}_{\text{function of states}} + \underbrace{J(u)}_{\text{input to output "coupling"}}$

Linear Systems State Model

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t) + D\,u(t)$$

Notice that A could be $A(t)$, this is called ~~the [illegible]~~. time variant.

Ex: Convert to the canonical form

- 1) $m\dddot{x}_1 + n x_1 = F(t)$ $\leftarrow$ 3rd order $\Rightarrow$ 3 states x_1 $\dot{x}_1$ $\ddot{x}_1$
  2) $m\dot{x}_2 + z x_1 = G(t)$ $\leftarrow$ 1st order $\Rightarrow$ 1 state x_2

$\underbrace{\text{3 states } x_1\ \dot{x}_1\ \ddot{x}_1 \quad \text{1 state } x_2}$ State vector $\Rightarrow$ $\begin{pmatrix} x_1 \\ \dot{x}_1 \\ \ddot{x}_1 \\ x_2 \end{pmatrix} = X$

- Solve 1) for $\dddot{x}_1$:

$$\dddot{x}_1 = \frac{F(t)}{m} - \frac{n x_1}{m}$$

and 2)

$$\dot{x}_2 = \frac{G(t)}{m} - \frac{z x_1}{m}$$

$$= \begin{bmatrix} -\frac{n}{m} & 0 & 0 & 0 \\ -\frac{z}{m} & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \\ \ddot{x}_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{F(t)}{m} \\ \frac{G(t)}{m} \end{pmatrix}$$

- Write time derivative of states

$$\dot{X} = \begin{pmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dddot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{n}{m} & 0 & 0 & 0 \\ -\frac{z}{m} & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \\ \ddot{x}_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{m} \end{bmatrix} \begin{pmatrix} u \end{pmatrix}$$

Laplace Transform of linear system model

$$\mathcal{L}(\dot{x}) = \mathcal{L}(Ax) + \mathcal{L}(Bu)$$
$$\mathcal{L}(y) = \mathcal{L}(Cx) + \mathcal{L}(Du)$$

define $\mathcal{L}(\underset{\text{case}}{\text{lower}}) = \underset{\text{case}}{\text{upper}}$

$$\mathcal{L}(x) = X$$
$$\mathcal{L}\square = \square$$

Identity $\mathcal{L}\left(\frac{d\square}{dt}\right) = s\square - \square(0)$

Thus, the time derivative is

$$sX - x(0) = AX + BU$$
$$Y = CX + DU$$

Solve for X in 1st eqn

$$sX - AX = BU + x(0) \Rightarrow X = (s-A)^{-1}(BU + x(0))$$

plug into 2nd eqn

$$Y = \underbrace{C(s-A)^{-1}BU + C(s-A)^{-1}x(0) + DU}$$

$$s = \begin{bmatrix} s & & \\ & s & \\ & & \ddots & \\ & & & s \end{bmatrix}$$

combine

$$Y = C(s-A)^{-1}x(0) + \left[C(s-A)^{-1}B + D\right]U$$

The transfer function is the ratio of gains of the $\mathcal{L}(\text{output}) / \mathcal{L}(\text{input})$

$$\boxed{G(s) = C(s-A)^{-1}B + D}$$

convert between state space and transfer function form.

Ex:

$$\dot{x} = \begin{bmatrix} -1 & 2 \\ 0 & -4 \end{bmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u$$

$$\Rightarrow G(s) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+1 & -2 \\ 0 & s+4 \end{bmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \frac{\begin{bmatrix} s+4 & 2 \\ 0 & s+1 \end{bmatrix}}{(s+1)(s+4)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0$$

Craymers Rule:
- flip diag
- neg off diag
- divide by det.

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & \frac{2}{(s+1)(s+4)} \\ & \frac{1}{s+4} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{s+1}$$

Thus the impulse response is

$$\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$$

why so simple?

Observability

Can the state vector be determined from the inputs and outputs over time?

Theorem: A LTI (linear time-invariant) system is observable iff

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{(n_x-1)} \end{bmatrix} \text{ has full rank}$$

Ex:

$$\dot{x} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix} x + \begin{bmatrix} B \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} D \end{bmatrix} u$$

$n_x = 2 \Rightarrow n_x - 1 = 1$

$\Rightarrow$

$$C = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 9 \\ 0 & -5 \end{bmatrix}$$

$$\mathcal{O} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 9 \\ 0 & -5 \end{bmatrix}$$ has at least 2 independent lines

$\boxed{\text{Observable}}$

Controllable

A system is controllable if the state can be transitioned from any state to any other state in a finite amount of time

Theorem: A LTI is controllable iff

$$\mathcal{C} = \begin{bmatrix} B & AB & A^{(n_x-1)}B \end{bmatrix} \text{ has rank } n_x$$

Ex: Above system with $B = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

$$AB = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 16 \\ 0 & 20 \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & -1 & 16 \\ 0 & 4 & 0 & 20 \end{bmatrix}$$ ✓ At least 2 independent columns.

$\boxed{\text{Controllable}}$

System Stability (Open loop)

Given an LTI system, the open loop response is stable iff the eigenvalues of A ~~are~~ have negative real parts

Ex: Is the following system stable?

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \qquad \Rightarrow A = \begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

Hint: We know that the sum of $\lambda_1 + \lambda_2 = -4$ (the trace $\equiv$ diagonal sum)

$$\det(\lambda I - A) = 0 \Rightarrow \begin{vmatrix} \lambda & -1 \\ 2 & \lambda+4 \end{vmatrix} = \lambda^2 + 4\lambda + 2 = 0 \Rightarrow \frac{-4 \pm \sqrt{16 - 4\cdot 1 \cdot 2}}{2}$$

$$= -2 \pm \frac{\sqrt{8}}{2}$$

$$= -2 \pm \frac{2\sqrt{2}}{2}$$

$$= 2\underbrace{\left(-1 \pm \frac{\sqrt{2}}{2}\right)}$$

-1 magnitude $>$ mag $\frac{\sqrt{2}}{2}$

$\lambda_1 \approx -0.58$

$\lambda_2 \approx -3.4$

$\boxed{\text{Stable}}$

Feed back Controller.

Design a system such that $u(t)$ is a function of states.

$$u = [K]\, x$$

often in literature as $u = -kx$

Substitute into LTI canonical form

$$\dot{x} = Ax + BKx \qquad \Rightarrow \qquad \dot{x} = [A + BK]\,x = \hat{A}\,x$$
$$y = Cx + DKx \qquad \qquad y = [C + DK]\,x = \hat{C}\,x$$

If we had anew impulse input w,

$$\dot{x} = \hat{A}x + \overbrace{\hat{B}}^{\text{Identity}} w$$
$$y = \hat{C}x$$

The impulse response is

$$g(t) = \mathcal{L}^{-1} G(s) = \mathcal{L}^{-1}\left(\hat{C}\,(s - \hat{A})^{-1}\hat{B}\right) = \mathcal{L}^{-1}[C + DK][sI - A - BK]^{-1}[I]$$

Is the system stable?

Is $\hat{A}$ stable? $\mathcal{R}(\mathrm{eig}(A)) \leq 0$

Ex: What value of the gain k_1 makes the following system unstable?

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad K = [k_1 \;\; 0]$$

1st: Is the system even controllable? $\mathcal{C} = [B \;\; AB] = \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \end{bmatrix}\right]$ ✓ yes.

Combined system

$$\hat{A} = A + BK = \begin{bmatrix} k_1 & 1 \\ k_1 - 2 & -3 \end{bmatrix} \Rightarrow \det(\lambda I - \hat{A}) = \det\begin{pmatrix} \lambda - k_1 & -1 \\ -k_1 + 2 & \lambda + 3 \end{pmatrix} = 0$$

Characteristic equ.

$$\lambda^2 + 3\lambda - k_1\lambda - 3k_1 + k_1 + 2 = 0$$

Rather than finding the roots (hard by hand), just recognize that we want the equivalent system $\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$ to have zero damping (i.e. $\zeta = 0$)

$$(+3 - k_1) = 0 \quad \Rightarrow \boxed{k_1 = 3}$$