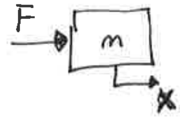
AEM 617

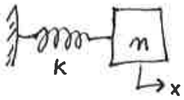
FCS Gov Eqns + Physics

Newton's Law

$$F = ma$$

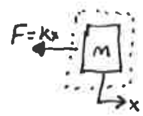


Ex: Develop the equations of motion for a mass-spring system



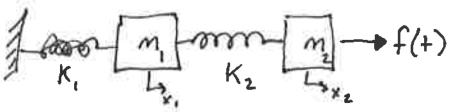
spring: $F = -Kx$

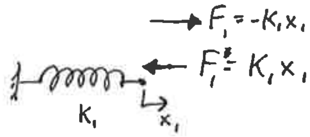
control volume around m

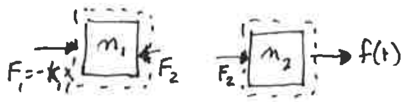


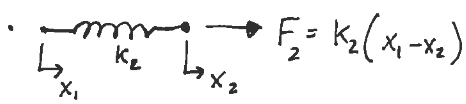
$$\Rightarrow \quad m\ddot{x} = -F = \cancel{kx} \quad \Rightarrow \quad \boxed{m\ddot{x} + kx = 0}$$

Ex: Develop for a 2 mass system









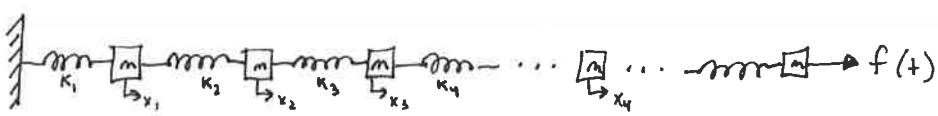
$$m_1\ddot{x}_1 = \underbrace{-K_1x_1}_{F_1} - K_2(x_1 - x_2)$$

$$m_2\ddot{x}_2 = F_2 = K_2(x_1 - x_2) + f(t)$$

$$\Rightarrow \quad \boxed{\begin{aligned} m_1\ddot{x}_1 + K_2(x_1 - x_2) + K_1x_1 &= 0 \\ m_2\ddot{x}_2 - K_2(x_1 - x_2) &= f(t) \end{aligned}}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\left(\ddot{X}\right) + \begin{bmatrix} K_2+K_1 & -K_2 \\ -K_2 & +K_2 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

Ex: Develop for an N mass system



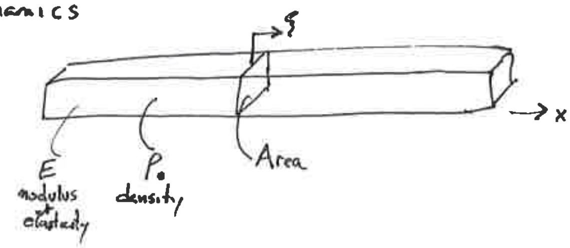
$$\begin{bmatrix} m_1 & & & & & \\ & m_2 & & & & \\ & & m_3 & & & \\ & & & \ddots & & \\ & & & & m_{N-1} & \\ & & & & & m_N \end{bmatrix}\left(\ddot{X}\right) + \begin{bmatrix} K_1+K_2 & -K_2 & & & \\ -K_2 & K_2+K_3 & -K_3 & & \\ & -K_3 & K_3+K_4 & \cdots & \\ & & \vdots & \ddots & \\ & & & & K_N \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \\ f(t) \end{pmatrix}$$

But what about more complicated cases?

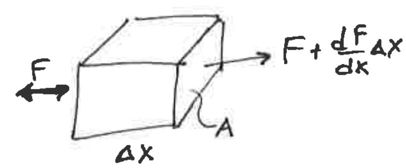
Solid Bar Systems

Statics: $F_1 = F_2$ Virtual work $F_1 \Delta x_1 = F_2 \Delta x_2$ with $\Delta x_1 = \Delta x_2$

Dynamics



on a Δx element



Summation of forces

$$F = ma \Rightarrow \Sigma F = \rho_0 A \Delta x \frac{d^2\xi}{dt^2} = -F + F + \frac{dF}{dx}\Delta x$$

Linear elasticity

$$\sigma = \frac{F}{A} \Rightarrow F = \sigma A = EA \frac{d\xi}{dx}$$

Substitute and remove Δx (generalize to any Δx)

$$\rho_0 A \frac{d^2\xi}{dt^2} = \frac{d}{dx}\left(EA \frac{d\xi}{dx}\right)$$

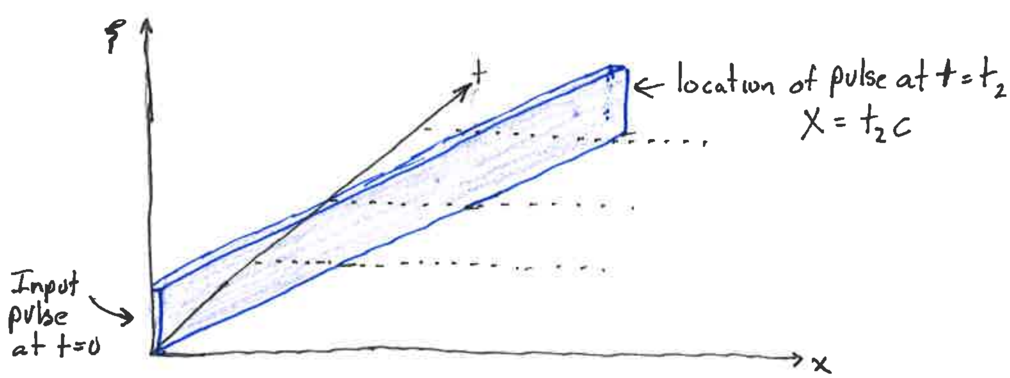
Assume EA is not varying with x

$$\rho_0 A \frac{d^2\xi}{dt^2} = EA \frac{d^2\xi}{dx^2} \Rightarrow \frac{d^2\xi}{dt^2} = \frac{EA}{\rho_0 A}\frac{d^2\xi}{dx^2}$$

This is a wave equation

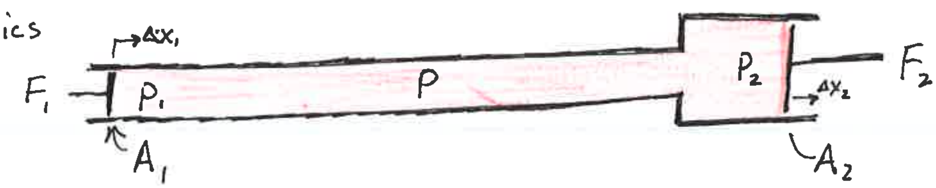
$$\xi_{tt} = \frac{E}{\rho_0}\xi_{xx} \qquad \text{where } c = \sqrt{\frac{E}{\rho_0}}$$

Information input at 1 takes a finite time to arrive at 2 $t = \frac{L}{c}$



Hydraulic Systems

Statics



Mass continuity and $P \cdot A = F$ imply

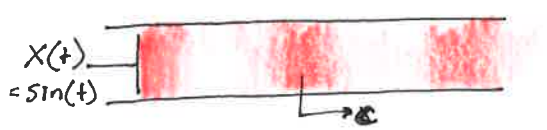
$$A_1 \Delta x_1 = A_2 \Delta x_2 \Rightarrow \frac{\Delta x_2}{\Delta x_1} = \frac{A_1}{A_2}$$

and

$$F_1 = P_1 A_1 \text{ and } F_2 = P_2 A_2 \quad \text{and} \quad P_1 = P_2 \Rightarrow \frac{F_1}{F_2} = \frac{A_1}{A_2}$$

such that $\frac{\Delta x_2}{\Delta x_1} = \frac{F_1}{F_2}$ or equal work $F_1 \Delta x_1 = F_2 \Delta x_2$

Dynamics



Fluids such as hydraulic fluid and water are compressible (although much less than air)

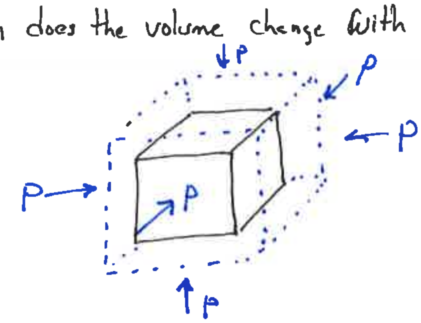
Thus, information (e.g. pressure) does not instantaneously arrive everywhere in the system at the same time.

Gov Equ: $\frac{\partial^2 P}{\partial t^2} = c^2 \nabla^2 P$ wave equation

Bulk Modulus

Given a volume of a material, how much does the volume change with pressure.

$$K = \rho \frac{dP}{d\rho} = V \frac{dP}{dV}$$



The speed of sound is

$$c = \sqrt{K/\rho} = \sqrt{\frac{dP}{d\rho}}$$

Unfortunately, in actual fluids, K and ρ vary with pressure and temperature.

For example, in distilled water (from Fundamentals of Acoustics, Kinsler, Frey, et. al.)

$$c\left[\tfrac{m}{s}\right] = 1402.7 + 488t - 482t^2 + 135t^3 + (15.9 + 2.8t + 2.4t^2)\left(\frac{P_{gage}}{100}\right)$$

$$t = \frac{T[C]}{100} \qquad P_{gage}\ [bar]$$

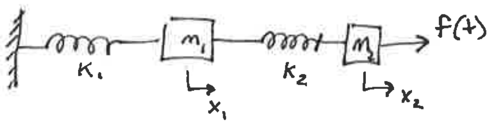
≈ 4200 ft/s at 0^{C} and static pressure.

Lagranges Equations

Given a N dimensional system (i.e. N degrees of freedom), with kinetic energy $T(\dot{q}_i, q, t)$ and potential energy $U = U(q_i, t)$ for a set of N coordinates q, and a generalized force Q_i in the q_i coordinate.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i \quad \text{for } i = 1, N$$

Ex:



$q_1 = x_1$

$q_2 = x_2$

$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$

$U = \frac{1}{2} K_1 x_1^2 + \frac{1}{2} K_2 (x_2 - x_1)^2$ ← notice that this is energy, so $(x_1 - x_2)^2 = (x_2 - x_1)^2$

$Q_2 = f(t)$ such that $Q_2 q_2$ is work.

$q_1 = x_1$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = \frac{d}{dt}\left(m_1 \dot{x}_1\right) = m_1 \ddot{x}_1$$

$$-\frac{\partial T}{\partial x_1} = 0$$

$$\frac{\partial U}{\partial q_1} = K_1 x_1 + \frac{1}{2} K_2 \cdot 2 \cdot (x_2 - x_1)(-1) = K_1 x_1 - K_2 (x_2 - x_1)$$

Combined:

$$\boxed{m_1 \ddot{x}_1 + K_1 x_1 - K_2 (x_2 - x_1) = 0}$$

$q_2 = x_2$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = \frac{d}{dt}\left(m_2 \dot{x}_2\right) = m_2 \ddot{x}_2$$

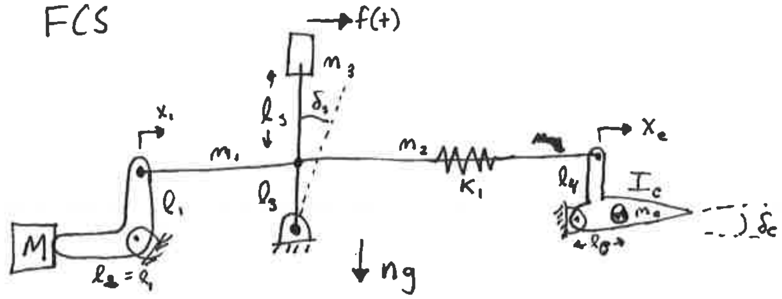
$$-\frac{\partial T}{\partial x_2} = 0$$

$$\frac{\partial U}{\partial x_2} = K_2 (x_2 - x_1)$$

$$Q_2 = f(t)$$

$$\boxed{m_2 \ddot{x}_2 + K_2 (x_2 - x_1) = f(t)}$$

Example FCS



States. δ_s and δ_e $\Rightarrow$ $x_1 = \ell_3 \delta_s$ $\quad x_e = \ell_4 \delta_e$

$$T = \underbrace{\frac{1}{2} M \left(\ell_3 \dot{\delta}_s\right)^2}_{\text{Inertial mass}} + \frac{1}{2} m_1 \left(\ell_3 \dot{\delta}_s\right)^2 + \frac{1}{2} m_2 \left(\ell_3 \dot{\delta}_s\right)^2 + \frac{1}{2} I_e \left(\overset{\dot{\delta}_e}{\cancel{\dot{x}_e/\ell_4}}\right)^2 + \frac{1}{2} m_3 \left(\ell_s \dot{\delta}_s\right)^2$$

$$U = \frac{1}{2} K_1 \left(\ell_3 \delta_s - \ell_4 \delta_e\right)^2$$

$$Q = \underbrace{\ell_s f(t)}_{\delta_s} + \underbrace{-Mng}_{\substack{\text{weight} \\ \delta_s}} + \underbrace{m_e \ell_e ng}_{\delta_e}$$

Gov Eqns from L'

δ_s:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\delta}_s}\right) = \frac{d}{dt}\left(M \ell_3^2 \dot{\delta}_s\right) + \frac{d}{dt}\left(m_1 \ell_3^2 \dot{\delta}_s\right) + \frac{d}{dt}\left(m_2 \ell_3^2 \dot{\delta}_s\right) + \frac{d}{dt}\left(m_3 \ell_s \dot{\delta}_s\right)$$

$$= \left(M \ell_3^2 \cancel{\delta} + m_1 \ell_3^2 + m_2 \ell_3^2\right) \ddot{\delta}_s$$

$+ m_3 \ell_s$

$$\frac{\partial U}{\partial \delta_s} = K_1 \left(\ell_3 \delta_s - \ell_4 \delta_e\right) \ell_3$$

$$Q_{\delta_s} = \ell_s f(t) - Mng$$

$$\Rightarrow \boxed{\left(M \ell_3^2 + m_1 \ell_3^2 + m_2 \ell_3^2 + m_3 \ell_s\right) \ddot{\delta}_s + K_1 \ell_3^2 \delta_s - K_1 \ell_3 \ell_4 \delta_e = \ell_s f(t) - Mng}$$

δ_e:

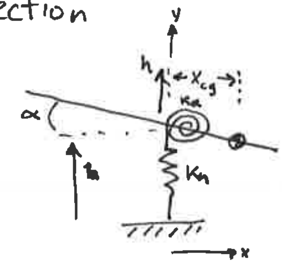
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\delta}_e}\right) = \frac{d}{dt}\left(I_e \dot{\delta}_e\right)$$

$$\frac{\partial U}{\partial \delta_e} = K_1 \left(\ell_3 \delta_s - \ell_4 \delta_e\right)(-\ell_4)$$

$$Q_{\delta_e} = m_e \ell_e ng$$

$$\Rightarrow \boxed{I_e \ddot{\delta}_e - K_1 \ell_4 \ell_3 \delta_s + \ell_4^2 \delta_e = m_e \ell_e ng}$$

Airfoil Section



at any location x, $y = h - x\sin\alpha$

$\dot{y} = \dot{h} - x\cos\alpha\,\dot{\alpha}$

Lagranges equ works for continuous forces ~~too~~. (as we will see)

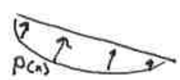
$$T = \int \frac{1}{2}\dot{y}^2\,dm = \frac{1}{2}\int\left(\dot{h} - x\cos\alpha\,\dot{\alpha}\right)^2 \rho\,dx$$

$$= \frac{1}{2}\int\left(\dot{h}^2 - 2\dot{h}x\cos\alpha\,\dot{\alpha} + x^2\cos^2\alpha\,\dot{\alpha}^2\right)\rho\,dx$$

$$= \frac{1}{2}\dot{h}^2\underbrace{\int \rho\,dx}_{\text{Mass}\equiv m} - \dot{h}\cos\alpha\,\dot{\alpha}\underbrace{\int x\rho\,dx}_{x_{cg}\cdot m\,\equiv\,S_\alpha} + \frac{1}{2}\cos^2\alpha\,\dot{\alpha}^2\underbrace{\int x^2\rho\,dx}_{\text{Rotational Inertia } I_\alpha}$$

$$U = \frac{1}{2}K_h h^2 + \frac{1}{2}K_\alpha \alpha^2$$

Virtual Work from aero forces $p(x)$



$$\delta W = \int p\,\delta y\,dx = \int p\left(\delta h - x\,\cancel{\sin}^{\,\delta\alpha}(\delta\alpha)\right)dx = \int p\,\delta h\,dx - \int p\cdot x\cdot\delta\alpha\,dx$$

$$= \underbrace{\int p\,dx}_{Q_{\delta h} = L}\,\delta h - \underbrace{\int p\cdot x\cdot dx}_{Q_{\delta\alpha} = M}\,\delta\alpha$$

Gov Equ.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{h}}\right) = \frac{d}{dt}\left(\dot{h}m - \cos\alpha\,\dot{\alpha}\,x_{cg}m\right)$$

$$\frac{\partial U}{\partial h} = K_h h \quad \text{and} \quad Q_{\delta_h} = L$$

$$\Rightarrow m\ddot{h} + \sin\alpha\,\dot{\alpha}^2 x_{cg}m - \cos\alpha\,\ddot{\alpha}\,x_{cg}m + K_h h = L$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\alpha}}\right) = \frac{d}{dt}\left(-\dot{h}\cos\alpha\,x_{cg}m + \cos^2\alpha\,\dot{\alpha}\,I_\alpha\right) = -\ddot{h}\cos\alpha\,x_{cg}m + \dot{h}\sin\alpha\,\dot{\alpha}\,x_{cg}m$$
$$+\cos^2\alpha\,\ddot{\alpha}\,I_\alpha - 2\cos\alpha\sin\alpha\,\ddot{\alpha}\,I_\alpha$$

$$\frac{\partial U}{\partial \alpha} = K_\alpha\alpha \qquad Q_{\delta_\alpha} = M$$

When α is small,

$$m\ddot{h} - S_\alpha \ddot{\alpha} + K_h h = L$$

$$-\ddot{h} S_\alpha + I_\alpha \ddot{\alpha} + K_\alpha \alpha = M$$

$$\begin{pmatrix} m & -S_\alpha \\ -S_\alpha & I_\alpha \end{pmatrix}\begin{pmatrix} \ddot{h} \\ \ddot{\alpha} \end{pmatrix} + \begin{bmatrix} K_h & 0 \\ 0 & K_\alpha \end{bmatrix}\begin{pmatrix} h \\ \alpha \end{pmatrix} = \begin{pmatrix} L \\ M \end{pmatrix}$$

Invert to seperate $\ddot{h}$ and $\ddot{\alpha}$

$$\begin{pmatrix} \ddot{h} \\ \ddot{\alpha} \end{pmatrix} + \frac{\begin{pmatrix} I_\alpha & S_\alpha \\ S_\alpha & m \end{pmatrix}}{I_\alpha m - S_\alpha^2}\begin{pmatrix} K_h & 0 \\ 0 & K_\alpha \end{pmatrix}\begin{pmatrix} h \\ \alpha \end{pmatrix} = \frac{\begin{pmatrix} I_\alpha & S_\alpha \\ S_\alpha & m \end{pmatrix}}{I_\alpha m - S_\alpha^2}\begin{pmatrix} L \\ M \end{pmatrix}$$

$$\begin{pmatrix} \ddot{h} \\ \ddot{\alpha} \end{pmatrix} + \frac{\begin{pmatrix} I_\alpha K_h & S_\alpha K_\alpha \\ K_h S_\alpha & K_\alpha m \end{pmatrix}}{I_\alpha m - S_\alpha^2}\begin{pmatrix} h \\ \alpha \end{pmatrix} = \frac{\begin{pmatrix} I_\alpha L + S_\alpha M \\ S_\alpha L + mM \end{pmatrix}}{I_\alpha m - S_\alpha^2}$$

Canonical 1st order

$$\frac{d}{dt}(\dot{h}) = \ddot{h}$$

$$\frac{d}{dt}\begin{pmatrix} h \\ \dot{h} \\ \alpha \\ \dot{\alpha} \end{pmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{I_\alpha K_h}{I_\alpha m - S_\alpha^2} & 0 & \frac{S_\alpha K_\alpha}{I_\alpha m - S_\alpha^2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K_h S_\alpha}{I_\alpha m - S_\alpha^2} & 0 & \frac{K_\alpha m}{I_\alpha m - S_\alpha^2} & 0 \end{bmatrix}\begin{pmatrix} h \\ \dot{h} \\ \alpha \\ \dot{\alpha} \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{I_\alpha}{I_\alpha m - S_\alpha^2} & \frac{S_\alpha}{I_\alpha m - S_\alpha^2} \\ 0 & 0 \\ \frac{S_\alpha}{I_\alpha m - S_\alpha^2} & \frac{m}{I_\alpha m - S_\alpha^2} \end{bmatrix}\begin{pmatrix} L \\ M \end{pmatrix}$$

Early FCS concepts/designs

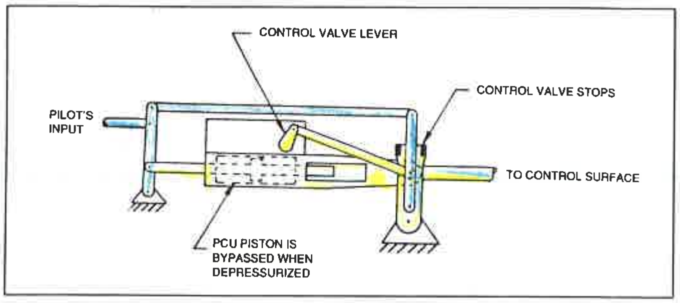
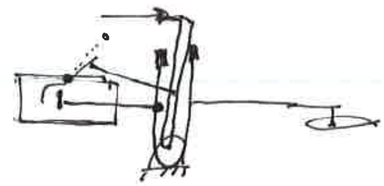


Figure 3-2. Stratoliner Power-Boost Actuator.



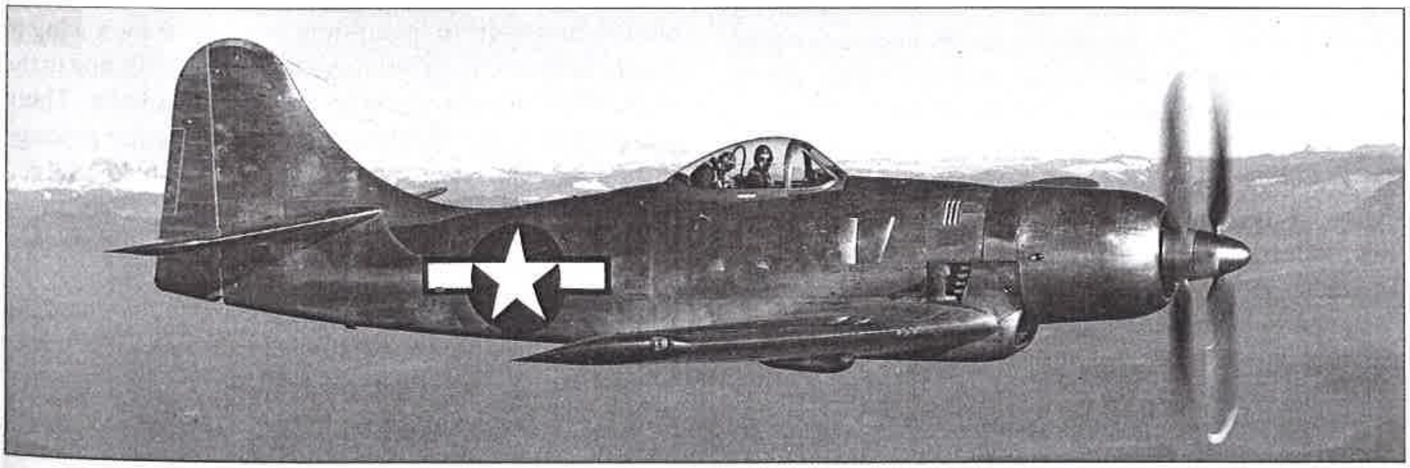


Figure 3-4. Boeing XF8B Prototype Navy Fighter Bomber, 1944.

450 kt
5 in 1

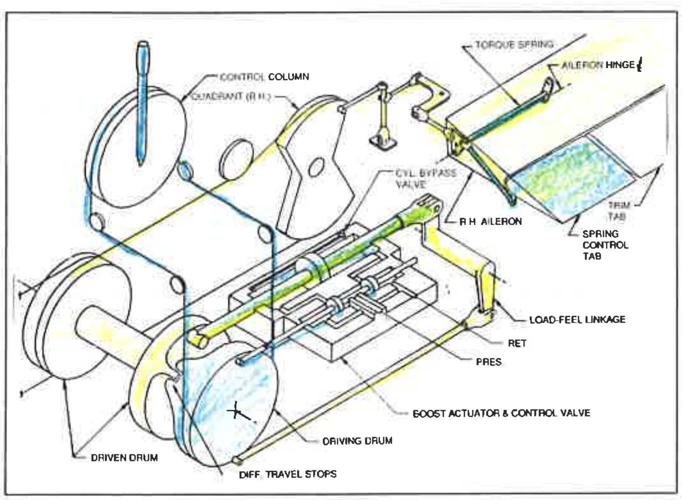
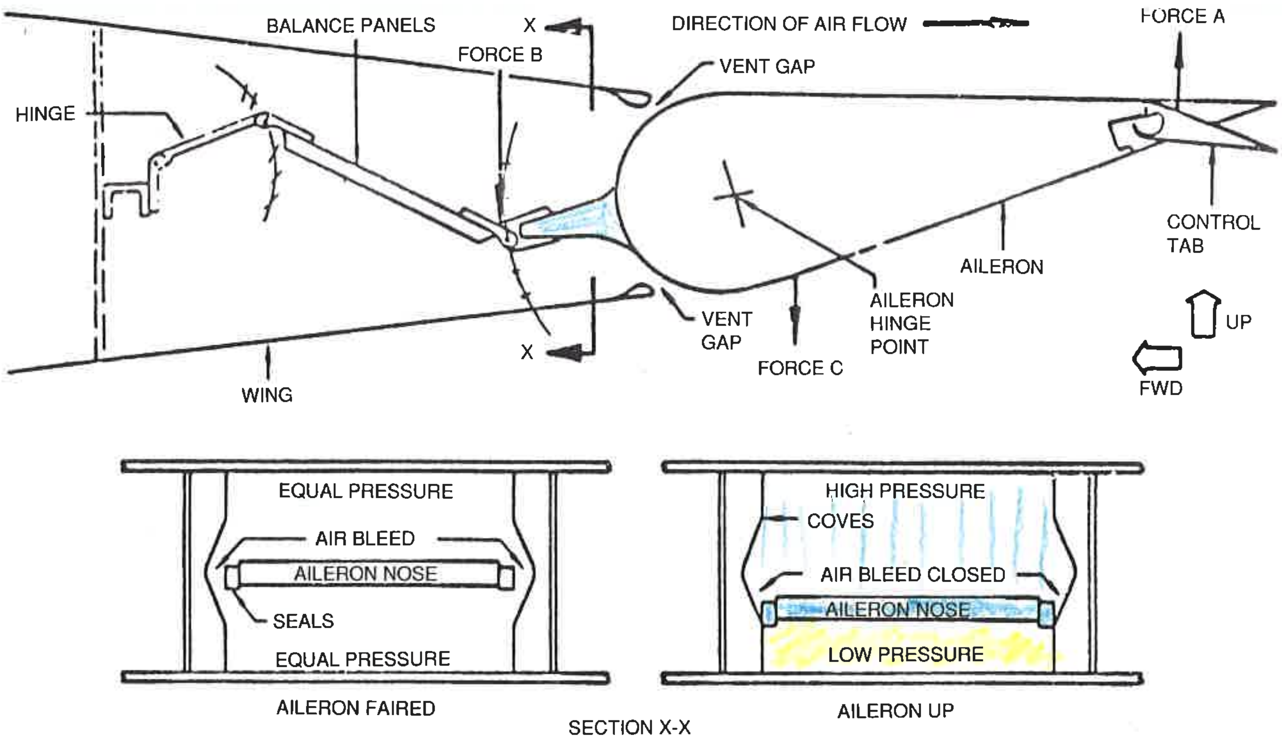


Figure 3-5. XF8B Aileron Power-Boost Actuation System.



1. Force A induced by air pressure differential on control tab initiates rotation of aileron.
2. Force B induced by air pressure differential on balance panels, after aileron is displaced by tab movement.
3. Force C as induced by air pressure differential on aileron, opposing forces A and B.
4. All forces are proportional to angular displacement of surfaces. For any given tab displacement, aileron rotates until the moment of force C is equal to the sum of the moments of forces A and B.
5. A controlled air bleed is provided at the ends of the aileron to proportion force B with displacement of aileron. See section X-X.

Figure 3-3. KC-135/707 Aileron Balance Panel Installation.

Source: Aircraft Flight Control Actuation System Design
Raymond + Chenoweth