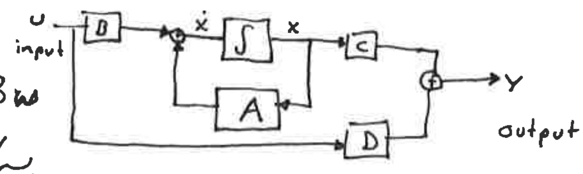
Kalman Filter

Estimate the states of a system given Known inputs and measurements

$$\dot{x} = Ax + Bu + Bw$$
$$y = Cx + Du + \underbrace{v}_{\text{noise!}}$$



Given y and u, can we determine x? optimally?

- Try simple algebra.

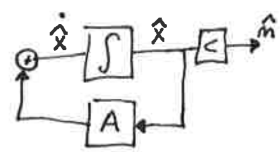
  If $y = Cx + Du \Rightarrow Cx = y - Du \Rightarrow x = C^{-1}y - C^{-1}Du$

  What is fragile about this estimate?

  - Both the measured output y and the input u could be noisy signals or could just be noisy actual values
  - How can we estimate x in the presence of NOISE?

    The same way you read the letters, 1) Smoothing 2) Fitting to expected shapes.

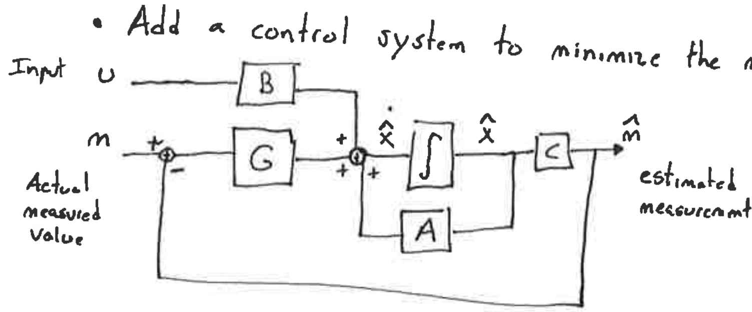
- Kalman Filter
  - Operate a state machine that seeks to optimally track outputs through an error correcting update.
  - Without inputs, we could mimic the actual system with state estimate $\hat{x}$



Need to know the system! [A]

$$\dot{\hat{x}} = A\hat{x}$$
$$\hat{m} = C\hat{x}$$

  - Add a control system to minimize the measurement error $(m - \hat{m})$



This is a Kalman Filter

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + G(m - C\hat{x})}$$

Error:

$$e = x - \hat{x} \equiv \text{actual value} - \text{estimated value of states}$$

$$\dot{e} = \dot{x} - \dot{\hat{x}}$$

$+Bw$

$$= Ax + Bu - A\hat{x} - Bu - G(m - C\hat{x})$$

why? noisy signals

$+Bw$

$$= Ax + Bu - A\hat{x} - Bu - G\,(Cx + Du + v\} - C\hat{x})$$

$$= A(x - \hat{x}) - GC(x - \hat{x}) + Bw - Gv$$

$$= Ae - GCe + [B \;\; G]\begin{pmatrix} w \\ v \end{pmatrix}$$

$$\dot{e} = (A - GC)\,e + [B \;\; G]\begin{pmatrix} w \\ v \end{pmatrix}$$

- Given that w and v are Gaussian noise with correlation S_w and S_v

$$E(w(t)\,w(t+\tau)) = \underbrace{S_w\,\delta(\tau)}_{\text{what does this mean?}}$$

- Result is the Ricatti Equation (since $\frac{d}{dt}(V_1 V_2) = V_1 \frac{dV_2}{dt} + \frac{dV_1}{dt} V_2$)

$$\dot{\Sigma}_e(t) = \Sigma_e A^T + A\Sigma_e + B_w S_w B_w^T - \Sigma_e C^T S_v^{-1} C \Sigma_e$$

such that

$$G(t) = \Sigma_e(t)\, C^T S_v^{-1}$$

and

$$\dot{\hat{x}}(t) = A\hat{x} + Bu + G(t)\left(m(t) - C\hat{x}\right)$$

- Comments
  - As the measurement noise v increases (i.e. S_v), the Gain G decreases.
  - As input noise increases (i.e. S_w), the gain increases! Why?

Steady State Kalman Filter (i.e. $G \neq G(t)$)

$\frown \dot{\Sigma}_e(t) = 0$

- Solve steady Riccati Equation (Matlab)

$$\Sigma_e A^T + A\Sigma_e + B_w S_w B_w^T - \Sigma_e C^T S_v^{-1} C \Sigma_e = 0$$

- Solve Hamiltonian Matrix (eigvectors) $(Av = \lambda v)$

$$\left[\begin{array}{c|c} -A^T & C^T S_v^{-1} C \\ \hline B_w S_w B_w^T & A \end{array}\right] \Psi = \Psi \Lambda$$

Then partition $\Psi = \left[\begin{array}{c|c} \Psi_{11} & \Psi_{12} \\ \hline \Psi_{21} & \Psi_{22} \end{array}\right] \Rightarrow \Sigma_e = \Psi_{22} (\Psi_{12}^{-1})$

$\frown G = \Psi_{22} \Psi_{12}^{-1} C^T S_v^{-1}$

More information

Linear Optimal Control, Jeffery Burl

Ex: Radar Tracking

We measure distance from time of flight.
Can we estimate velocity too?

$$\begin{pmatrix}\dot r\\ \ddot r\end{pmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{pmatrix} r\\ \dot r\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} w \qquad \Rightarrow \qquad A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \quad B = \begin{pmatrix}0\\1\end{pmatrix}$$

$$m = \begin{bmatrix}1 & 0\end{bmatrix}\begin{pmatrix} r\\ \dot r\end{pmatrix} + v \qquad C = \begin{bmatrix}1 & 0\end{bmatrix} \quad D = \begin{bmatrix}1\end{bmatrix}$$

If the std-dev of range error is 100 ft, then $S_v = \sigma^2 \Delta t$

Say, the scanning time is 1 second

$$S_v = (100)^2 \cdot 1 = 10000 \text{ ft}^2 \text{ s}$$

Also assume that the air is glassy smooth, the std-dev ~~variance~~ of flight acceleration speed is $1 \frac{ft}{s^2}$! $S_w = 1 \frac{ft^2}{s^4}\frac{1}{Hz}$

The Hamiltonian is

$$\begin{bmatrix}0 & 0\\ -1 & 0\end{bmatrix} \quad \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}10000\end{bmatrix}^{-1}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}\frac{1}{10000} & 0\\ 0 & 0\end{bmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix}(1)\begin{pmatrix}0 & 1\end{pmatrix} \quad \cancel{\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}} \quad \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

Eigen~~values~~ vectors are

−0.0007	−0.0007
0	0
0.99499	0.99499
−0.07036	−0.07036

~~Yuck~~ Yuck!, Not numerically stable!

Iterate Riccatti equ in time (until $\dot\Sigma_e = 0$)

$$\Sigma = \begin{bmatrix}1414 & 100\\ 100 & 14.14\end{bmatrix} \Rightarrow G = \begin{bmatrix}1414 & 100\\ 100 & 14.14\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\frac{1}{10000} = \begin{pmatrix}0.1414\\ 0.01\end{pmatrix}$$

$$\dot{\hat x} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\hat x + \begin{pmatrix}0\\1\end{pmatrix}u + \begin{pmatrix}0.1414\\ 0.01\end{pmatrix}\left(m - \begin{bmatrix}1 & 0\end{bmatrix}\hat x\right)$$