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STEP RESPONSES OF SECOND ORDER SYSTEMS

Second order systems with a step input have the general equation,

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y = U$$

where U is the step size. The characteristic equation of a 2nd order system is,

$$r^2 + 2\zeta\omega_n r + \omega_n^2 = 0$$

which contains the roots,

$$r_1, r_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

The roots (r_1, r_2) of the characteristic equation give the coefficients of the general homogeneous solution. For step inputs, the particular solution is $y_p = \frac{U}{\omega_n^2}$. Thus, the total solution is,

$$y(t) = \frac{U}{\omega_n^2} + K_1e^{r_1t} + K_2e^{r_2t}$$

Over Damped

Overdamped systems occur when the roots of the characteristic equation are real. This occurs when $\zeta > 1$. The solution to the 2nd order system equation is

$$y(t) = y_p + y_h$$
$$y(t) = \frac{U}{\omega_n^2} + K_1e^{r_1t} + K_2e^{r_2t}$$

Critically Damped

When the roots of the characteristic equation are real and equal, the system is critically damped. This occurs when $\zeta = 1$.

$$y(t) = y_p + y_h$$
$$y(t) = \frac{U}{\omega_n^2} + K_1e^{r_1t} + K_2te^{r_1t}$$

Underdamped

Underdamped systems occur when the characteristic equation has imaginary components. This occurs when $0 < \zeta < 1$.

$$y(t) = y_p + y_h$$
$$y(t) = \frac{U}{\omega_n^2} + K_1e^{r_1t} + K_2te^{r_2t}$$

Both r_1 and r_2 contain imaginary parts. Since the solution must contain only real parts, Euler's formula is applied.

$$e^{\jmath\theta} = cos(\theta) + \jmath sin(\theta)$$
$$e^{-\jmath\theta} = cos(\theta) - \jmath sin(\theta)$$

Combining Euler's formula and the general solution for 2nd order systems yields,

$$y(t) = \frac{U}{\omega_n^2} + K_1e^{-\zeta\omega_nt}cos(\omega_n\sqrt{(1-\zeta^2)}) + K_2e^{-\zeta\omega_nt}sin(\omega_n\sqrt{(1-\zeta^2)})$$