Simpson Integration Rules
Fast Derivation with B-Splines
(including partial integrals)

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Abstract

The 3 and 4 point Simpson rules numerically integrate equally-spaced discrete data values. The 4 point rule is typically named Simpson's 3/8 rule. This short note describes an interesting and fast derivation of Simpson's rules including the associated partial integrals. Non-multiple data point integration is discussed in a surprising finale to Simpson's 3/8ths rule.

Simpson Rule

Deriving the time integration follows the traditional method of exactly integrating a known 2nd order function for arbitrary coefficients. Second order integration uses three data points, conveniently represented in this derivation by the three coefficient b-spline.

$$\phi(t) = \left[\begin{array}{ccc} (1-t)^2 & 2t\,(1-t) & t^2 \end{array} \right]$$

Exactly integrating the basis ϕ gives definite integrals in terms of an arbitrary coefficient vector a. In matrix form, the definite integral

$$r_t - r_1 = I_t = \int_{t_1}^{t} \phi(s)\, ds$$

reduces to

$$\left(\begin{array}{c} I_2 \\ I_3 \end{array} \right) = \left[\begin{array}{ccc} \frac{7}{24} & \frac{1}{6} & \frac{1}{24} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array} \right] \left(\begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right)$$

The matrix representation to convert from coefficients to values is

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Inverting gives the required matrix representation for converting values to coefficients

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Combining the integral and conversion matrices produces the definite integrals in terms of function values

$$\begin{pmatrix} I_2 \\ I_3 \end{pmatrix} = \begin{bmatrix} \frac{7}{24} & \frac{1}{6} & \frac{1}{24} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

This traditional Simpson rule integrator from t_1 to t_3 is

$$r_3 - r_1 = h\left(\frac{1}{6}U_1 + \frac{2}{3}U_2 + \frac{1}{6}U_3\right)$$

The partial integral t_1 to t_2 rule is

$$r_2 - r_1 = h\left(\frac{5}{24}U_1 + \frac{1}{3}U_2 - \frac{1}{24}U_3\right)$$

By subtraction, the partial integral t_2 to t_3 rule is

$$r_3 - r_2 = h\left(-\frac{1}{24}U_1 + \frac{1}{3}U_2 + \frac{5}{24}U_3\right)$$

Simpson's 3/8 Rule

Proceeding as before, we now pick a four coefficient basis

$$\phi(t) = \begin{bmatrix} (1-t)^3 \\ 3t\,(1-t)^2 \\ 3t^2\,(1-t) \\ t^3 \end{bmatrix}$$

In matrix form, the definite integral

$$r_t - r_1 = I_t = \int_{t_1}^{t} \phi(s)\, ds$$

reduces to

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{65}{324} & \frac{11}{108} & \frac{1}{36} & \frac{1}{324} \\ \frac{20}{81} & \frac{2}{9} & \frac{4}{27} & \frac{4}{81} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

The matrix representation to convert from coefficients to equally spaced values, $t = \left[0, \frac{1}{3}, \frac{2}{3}, 1\right]$, is

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{8}{27} & \frac{4}{9} & \frac{2}{9} & \frac{1}{27} \\ \frac{1}{27} & \frac{2}{9} & \frac{4}{9} & \frac{8}{27} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

Inverting gives the required matrix representation for converting values to coefficients

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{5}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\ \frac{1}{3} & -\frac{3}{2} & 3 & -\frac{5}{6} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

Combining the integral and conversion matrices produces the definite integrals in terms of function values

$$\begin{aligned} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{65}{324} & \frac{11}{108} & \frac{1}{36} & \frac{1}{324} \\ \frac{20}{81} & \frac{2}{9} & \frac{4}{27} & \frac{4}{81} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{5}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\ \frac{1}{3} & -\frac{3}{2} & 3 & -\frac{5}{6} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{8} & \frac{19}{72} & -\frac{5}{72} & \frac{1}{72} \\ \frac{1}{9} & \frac{4}{9} & \frac{1}{9} & 0 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \end{aligned}$$

The familiar Simpson 3/8 rule is recovered on the bottom row.

$$I = h\left(\frac{1}{8}U_1 + \frac{3}{8}U_2 + \frac{3}{8}U_3 + \frac{1}{8}U_4\right)$$

Interestingly, the 3rd row contains a partial integral up to the third point (at 2/3) with zero contribution from the 4th point. Stretched to an equivalent stepsize, the 3rd row rule is exactly the previously derived 3 point Simpson rule! Knapsacking these methods as advised in most numerical methods classes actually has merit.