Thin Airfoil Theory

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ABSTRACT

The thin airfoil theory for calculation of section flight properties is reviewed. Lift and moment coefficient and center of pressure calculations are made for cambered and flapped wing sections. Effects of camber and flaps are discussed. The thin wing theory results are compared to experimental airfoil data.

INTRODUCTION

Aircraft fly by overcoming gravity with a lifting force. This force can be provided by a wing. A wing's cross sectional geometry influences the flow of air. This combined geometry of the wing and the reaction of the air causes any general solution of wing section properties to become too complicated to use or impossible to find.

A more simple approach to finding flight properties of wing sections is to assume an inviscid and incompressible flow. A vortex superimposed in a airstream simulates the process of lift generation by a wing section. A vortex distribution placed along the wing will simulate the actual properties of the wing and allow a simple method of calculating wing properties. The thin airfoil theory calculates a distribution of vortices that is compatible with a thin representation of an airfoil. This distribution can be used to find the lift, moment and pressure properties of an airfoil.

The objective is to review the thin airfoil theory and to apply the theory to three wing sections. The wing sections will consist of one curved cambered NACA 2412 airfoil, one straight line flapped airfoil and one cambered line segment airfoil. The coefficients of lift and moment and the center of pressure locations will be found.

THEORY

The thin airfoil theory simulates the aerodynamic properties of an airfoil section with vortex sheets. The vortex sheet consists of a continuous vortex distribution along the chord of the wing. The induced velocity at a distance r caused by a vortex of strength $\gamma$ is

$$dv = \frac{\gamma ds}{2\pi r}$$

Integrating over the wing chord yields the total downward velocity due to the vortex sheet.

$$v = \int_{chord} \frac{\gamma ds}{2\pi r}$$

Because fluid cannot flow through the wings surface, the downward velocity at the wing's

surface must be zero. Thus a thin wing section at an small angle of attack can be approximated by

$$\frac{v(x)}{U} = \alpha - \frac{dz}{dx}$$

where U is the flight velocity and $\frac{dz}{dx}$ is the slope of the wing section with respect to the chord line. Substituting for the downward velocity, where $r = x_0 - x$, yields,

$$\frac{1}{2\pi U} \int_{chord} \frac{\gamma(x)}{x_0 - x} dx = \alpha - \frac{dz}{dx}$$

Additionally, the Kutta condition forces a fluid to flow off a surface tangentailly causing the vortex strength to be zero at the trailing edge. For ease of later calculation, the coordinate system is transformed to a polar system centered at the mid chord by $x = \frac{c}{2}(1 - \cos(\theta))$. Thus,

$$\frac{1}{2\pi U} \int_0^\pi \frac{\gamma(\theta) \sin(\theta)}{\cos(\theta) - \cos(\theta_0)} d\theta = \alpha - \frac{dz}{dx}$$

A general solution of the above equation is unknown and would be in any case too complex to handle. However, an approximation can be found by assuming a distribution of $\gamma$,

$$\gamma(\theta) = 2U \left[ A_0 \frac{1 + \cos(\theta)}{\sin(\theta)} + \sum_{n=1}^{\infty} A_n \sin(n\theta) \right]$$

Where values of $A_0$ through $A_n$ are given by (McCormick, 1995)

$$A_0 = \alpha - \frac{1}{\pi} \int_0^\pi \frac{dz}{dx} d\theta$$

$$A_n = \frac{2}{\pi} \int_0^\pi \frac{dz}{dx} \cos(n\theta) d\theta$$

Applying the Jutta Joukowski expression for lift and moment versus vortex strength yields,

$$C_l = 2\pi A_0 + \pi A_1$$

$$C_{m_{LE}} = -\frac{\pi}{2}(A_0 + A_1 - \frac{A_2}{2})$$

The moment at any arbitrary point off the leading edge can be found by adding the contribution due to the lifting force and the distance from the leading edge. At the quarter

chord location, the resulting moment does not depend on the angle of attack and is given by (McCormick, 1995),

$$C_{m_{0.25c}} = -\frac{\pi}{4}(A_1 - A_2)$$

Similarily, the center of pressure is given by

$$x_{cp} = \frac{c}{4}\left(1 + \frac{\pi}{C_l}(A_1 - A_2)\right)$$

## RESULTS AND DISCUSSION

Calculations were performed and are given in Appendix A, B and C. Three profiles were analized as thin wings. These included one cambered wing consisting of constantly changing curvature along a mean chord line. One flapped and one cambered wing both consisting of straight line segments.

The NACA 2412 cross section is given in Appendix A. The forward section $\frac{dz}{dz}$ as calculated from the mean camber line is $0.125\cos(\theta) - 0.025$. The aft section has a $\frac{dz}{dz}$ of $0.0555\cos(\theta) - 0.0111$. The mean camber line equations are switched at $\theta = 1.369$radians. Thus, $A_0 = \alpha - 0.0045165$. $A_1 = 0.0814604$ and $A_2 = 0.0138724$. The coefficient of lift is $C_l = 2\pi\alpha + 0.2275$ and the moment coefficient at the quarter chord is $C_{m_{0.25c}} = -0.05308$. The wing section has zero lift at an angle of attack of $-2.07$ degrees.

The flapped airfoil is given in Appendix B. The forward non-flapped section's $\frac{dz}{dx}$ is $0.086135$ and the aft flapped section has a $\frac{dz}{dx}$ of $-0.3655$. The wing switches from unflapped to flapped segments at $\theta$ equals $2.2377$ radians. Thus, $A_0 = \alpha + 0.04381$. $A_1 = 0.22591$ and $A_2$ equals $-0.013974$. The effective coefficient of lift is $C_{l_{eff}} = 2\pi\alpha + 0.9850$ and the effective moment coefficient at the quarter chord is $C_{m_{0.25c_{eff}}} = -0.28718$. The effective center of pressure is at $x_{cp} = \frac{c}{4}\left(1 + \frac{1.14903}{2\pi\alpha_{eff}+0.9850}\right)$ When converted to true chord lengths and true angles of attack, $C_l = 0.9849 \cdot 2\pi + 1.5018$, $C_{m_{ac}} = -0.27858$ and $x_{cp} = 0.9849\frac{c}{4}\left(1 + \frac{1.14903}{2\pi\alpha+0.9850}\right)$

The cambered airfoil is given in Appendix C. The forward section's $\frac{dz}{dx} = 0.8$ and the aft section has a $\frac{dz}{dx}$ of $-0.8$. The wing switches profile lines at $\theta = 90degrees$. $A_0$ is simply

$\alpha$. $A_1 = 0.10186$ and $A_2$ equals 0. At 10 degrees angle of attack, the coefficient of lift is $C_l = 1.4166$ and the moment coefficient at the quarter chord is $C_{m_{0.25c}} = -0.08$.

In each case, the coefficient of lift depends on a the angle of attack plus an increment due to the curvature of the wing. For the flapped wing, the increment was 1.58 while the unflapped NACA and cambered wing had an increment of approximately 0.3. This is consistent with flap increasing the lift at a constant true angle of attack.

Abbott and Doenhoff give experimental data for the NACA 2412. The experimental coefficient of lift at zero angle of attack is approximately 0.25 which is close to 0.228 as given by the thin wing theory. The increase of the coefficient of lift versus angle of attack shown experimentally is approximately 0.12 per degree. The thin wing gives $\frac{dC_l}{d\alpha} = 2\pi$ or 0.11 per degree. Both experimental and theoretical values of the zero lift angle of attack are approximately $-2$ degrees. The similarity of the experimental and theoretical values is remarkable considering the assumptions and simple mathematics of the thin wing theory.

## CONCLUSION

The thin airfoil theory is a method of calculating wing section properties. The thin wing theory only requires an expression of the mean chord line and thus can handle flapped and continuous wings. The mathematics are simple and involve only at most integration and differentiation. The results seem to be accurate when compared to experimental data when restricted to moderate angles of attack. The thin wing theory seems to be an adequate method for quick, simple and accurate flight properties of a wing section.

REFERENCES

McCormick, Barnes W., (1995)

*Aerodynamics, aeronautics, and flight mechanics*, John Wiley & Sons, New York.

Abbott, Ira H., Doenhoff, Albert E. (1959)

*Theory of Wing Sections*, Dover Publications, New York.

APPENDIX A
Calculations: NACA 2412

NACA 2412:

The NACA 2412's mean camber line is described piecewise with a switch in equations at x=0.4c.

$$Front: \quad \frac{z}{c} = 0.125(0.8\frac{x}{c} - \frac{x}{c}^2)$$

$$Aft: \quad \frac{z}{c} = 0.0555(0.2 + 0.8\frac{x}{c} - \frac{x}{c}^2)$$

The derivatives of z with respect to x are,

$$\frac{dz}{dz}_f = \frac{-0.25(x - 0.4c)c}{c^2} = -0.25\frac{x}{c} + 0.1$$

$$\frac{dz}{dz}_a = \frac{-0.111(x - 0.4c)c}{c^2} = -0.111\frac{x}{c} + 0.0444$$

The slopes are transformed into polar coordinates by substituting $x = \frac{c}{2}(1 - \cos(\theta))$ into the above derivatives. Thus,

$$\frac{dz}{dz}_f = -0.25 \cdot 0.5(1 - \cos(\theta)) + 0.1 = 0.125\cos(\theta) - 0.025$$

$$\frac{dz}{dz}_a = -0.111 \cdot 0.5(1 - \cos(\theta)) + 0.0444 = 0.0555\cos(\theta) - 0.0111$$

Also, the switching point is transformed. $x = 0.4c = 0.5c(1 - \cos(\theta))$. Thus the angle for switching equations is 1.369 radians.

From theory, $A_0 = \alpha - \frac{1}{\pi}\int_0^\pi \frac{dz}{dx}d\theta$ and $A_n = \frac{2}{\pi}\int_0^\pi \frac{dz}{dx}\cos(n\theta)d\theta$. The integration takes into account the location and distance of the front and aft equation for the section. Thus,

$$A_0 = \alpha - \frac{1}{\pi}\int_0^{1.369}(0.125\cos(\theta) - 0.025)d\theta \quad - \frac{1}{\pi}\int_{1.369}^\pi (0.0555\cos(\theta) - 0.0111)d\theta$$

$$= \alpha - 0.0280872 - (-0.0235707) = \alpha - 0.0045165$$

$$A_1 = \frac{2}{\pi}\int_0^{1.369}(0.125\cos(\theta) - 0.025)\cos(\theta)d\theta \quad + \frac{2}{\pi}\int_{1.369}^\pi (0.0555\cos(\theta) - 0.0111)\cos(\theta)d\theta$$

$$= 0.0466913 + 0.0347691 = 0.0814604$$

$$A_2 = \frac{2}{\pi}\int_0^{1.369}(0.125\cos(\theta) - 0.025)\cos(2\theta)d\theta \quad + \frac{2}{\pi}\int_{1.369}^\pi (0.0555\cos(\theta) - 0.0111)\cos(2\theta)d\theta$$

$$= 0.0249503 - 0.0110779 = 0.0138724$$

From theory, the lift coefficient is $C_l = 2\pi A_0 + \pi A_1$ and the moment coefficient about the quarter chord is $C_{m_{0.25c}} = -\frac{\pi}{4}(A_1 - A_2)$. Thus, the lift and moment coefficients are

$$C_l = 2\pi(\alpha - 0.0045165) + 0.0814604\pi = 2\pi\alpha + 0.2275$$

$$C_{m_{0.25c}} = -\frac{\pi}{4}(0.0814604 - 0.0138724) = -0.05308$$

The section has zero lift when $C_l$ equals zero. Solved for alpha, the zero lift angle is

$$\alpha = -\frac{0.2297}{2\pi} = -0.0362 radian = -2.07 degrees$$

Thus, the angle of attack for zero lift is -2.07 degrees.

APPENDIX B
Calculations: Flapped airfoil

Flapped airfoil:

The thin airfoil shown above has a 20 percent flap deflected 25 degrees. The effective chord length is found from the law of cosines, $c^2 = A^2 + B^2 - 2AB\cos(c) = 0.8^2 + 0.2^2 - 2 \cdot 0.8 \cdot 0.2 \cos(180 - 25)$ which yields an effective chord length of 0.9849 of the original chord. From the law of sines, the angle between the front section and the chord line is found.

$$\frac{sin(\beta)}{0.2} = \frac{sin(155)}{0.9849}$$

Thus, $\beta = 4.923 degrees$ and $\delta = 20.077 degrees$. The flap width projected along the effective chord line is $0.2c\sin(20.077) = 0.1878c$ or $0.190726c_{eff}$ when in terms of the effective chord length. So, the effective chord length of the unflapped section of the wing is $0.80927c_{eff}$.

The slope of the lines forming the wing and flap are

$$\frac{dz}{dx}_f = \tan(4.923) = 0.0861346$$

$$\frac{dz}{dx}_a = -\tan(20.077) = -0.3655$$

The switch from the forward to the aft equation is at $x = 0.8092736 = 0.5(1 - \cos(\theta))$ which is at 2.2377 radians in the polar coordinate system.

From theory, the A coefficients are,

$$A_0 = \alpha - \frac{1}{\pi}\int_0^{2.2377} 0.0861346 d\theta \quad - \frac{1}{\pi}\int_{2.2377}^{\pi} -0.3655 d\theta$$

$$= \alpha - 0.0613521 + 0.1051609 = \alpha + 0.04381$$

$$A_1 = \frac{2}{\pi}\int_0^{2.2377} 0.0861346\cos(\theta)d\theta \quad + \frac{2}{\pi}\int_{2.2377}^{\pi} -0.3655\cos(\theta)d\theta$$

$$= 0.0430861 + .1828297 = 0.22591$$

$$A_2 = \frac{2}{\pi}\int_0^{2.2377} 0.0861346\cos(2\theta)d\theta \quad + \frac{2}{\pi}\int_{2.2377}^{\pi} -0.3655\cos(2\theta)d\theta$$

$$= -0.0266512 - 0.1130904 = -0.13974$$

The effective lift and moment coefficients are

$$C_{l_{eff}} = 2\pi(\alpha + 0.04381) + 0.22591\pi = 2\pi\alpha + 0.9850$$

$$C_{m_{0.25c_{eff}}} = -\frac{\pi}{4}(0.2259158 - (-0.1397416)) = -0.28718$$

The location of the center of pressure with respect to the effective chord length is

$$x_{cp} = \frac{c}{4}(1 + \frac{\pi}{C_l}(A_1 - A_2)) = \frac{c}{4}\left(1 + \frac{1.14903}{2\pi\alpha_{eff} + 0.9850}\right)$$

The lift and moment coefficients and the center of pressure can be converted to the true chord length and angle of attack.

$$\alpha = \alpha_{eff} - 4.923°$$

$$c = \frac{c_{eff}}{0.9849}$$

Substituting the true values for the effective values and then simplifying yields,

$$C_l = C_{l_{eff}}(\frac{c_{eff}}{c}) = 0.9849 \cdot (2\pi(\alpha + 4.923°) + 0.9850) = 0.9849 \cdot 2\pi + 1.5018$$

$$C_{m_{ac}} = C_{m_{0.25c_{eff}}} \cdot (\frac{c_{eff}}{c})^2 = -0.27858(0.9849)^2 = -0.27858$$

$$x_{cp} = x_{cp_{eff}} \cdot (\frac{c_{eff}}{c}) = 0.9849\frac{c}{4}\left(1 + \frac{1.14903}{2\pi\alpha + 0.9850}\right)$$

APPENDIX C
Calculations: Thin Cambered Airfoil

Thin Cambered Airfoil:

The slope of the front piece of the airfoil is found from the thickness and half chord length. $\frac{dz}{dx} = \frac{0.04}{0.5} = 0.8$ The aft piece has the negative magnitude of the front slope. So, $\frac{dz}{dx} = \frac{-0.04}{0.5} = -0.8$ Transforming the coordinates yields 90 degrees ($\frac{\pi}{2}$ radians) as the location to switch slope equations. Thus,

$$A_0 = \alpha - \frac{1}{\pi}\int_0^{0.5\pi} 0.08 d\theta \quad - \frac{1}{\pi}\int_{0.5\pi}^{\pi} -0.08 d\theta$$

$$= \alpha - 0.04 - (0.04) = \alpha$$

$$A_1 = \frac{2}{\pi}\int_0^{0.5\pi} 0.08\cos(\theta)d\theta \quad + \frac{2}{\pi}\int_{0.5\pi}^{\pi} -0.08\cos(\theta)d\theta$$

$$= 0.0509296 + 0.0509296 = 0.1018592$$

$$A_2 = \frac{2}{\pi}\int_0^{0.5\pi} 0.08\cos(2\theta)d\theta \quad + \frac{2}{\pi}\int_{0.5\pi}^{\pi} -0.08\cos(2\theta)d\theta$$

$$= 0 + 0 = 0$$

The coefficient of lift is

$$C_l = 2\pi\alpha + 0.1018592\pi = 2\pi\alpha + 0.320$$

The quarter chord moment coefficient is

$$C_{m_{0.25c}} = -\frac{\pi}{4}(0.1018592 - 0) = -0.08$$

At 10 degrees (0.1745 radians) angle of attack as shown, the lift and moment coefficients are,

$$C_l = 2\pi \cdot 0.1745 + .320 = 1.4166$$

$$C_{m_{0.25c}} = -0.08$$