Antenna Wire Vibration
MAE 5093

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Abstract

A finite difference one-dimensional wave equation solver was developed to solve forced and free vibrations of an antenna wire. Four testcases are presented to determine the vibration properties and to visualize the resulting mode shapes. Conclusions are made concerning the applicability of the numerical method.

Introduction

The objective is to computationally solve the transverse vibration of a low-frequency antenna wire. The fundamental governing equation will be the one-dimensional wave equation for strings.

First, wave equation theory will be reviewed. Then, the computer program will be discussed. Next, results will be presented. Finally, conclusion will be made.

Theory

Wave Equation

The wave equation describes many physical systems. These systems include acoustic wave propagation, string vibrations, axial rod vibrations, drumhead vibrations and others. The general wave equation is hyperbolic partial differential equations of at least two variables. A representative form is given below.

$$\nabla^2 \Phi = C \frac{\partial^2 \Phi}{\partial t^2}$$

For taut string vibrations, the governing one-dimensional wave equation [2] is derived from a force balance along a segment of the string. The governing equation is given below.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}$$

The dimensional frequency term, $1/c^2$, is determined by the string's tension and unit mass.

$$c = \sqrt{T/\rho}$$

Exact Solution

From theory, it can be shown [2] that any function satisfies the governing equation provided that the function consists of a traveling wave. Thus, any function of the following form holds!

$$y(x,t) = A \cdot f(x - ct) + B \cdot f(x + ct)$$

For harmonic solutions, the traveling wave solution is commonly transformed to a standing wave solution. The following expression shows a standing wave solution.

$$y(x,t) = A \cdot \sin\left(n \cdot \pi \cdot \frac{x}{L}\right) \cdot \cos\left(n \cdot \pi \cdot c \cdot \frac{t}{L}\right)$$

Because the system is linear, an arbitrary number of solutions also hold. The following Fourier series allows an arbitrary function to be expressed as a sum of harmonics.

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(n \cdot \pi \cdot \frac{x}{L}\right) \cos\left(n \cdot \pi \cdot c \cdot \frac{t}{L}\right)$$

Finite Difference Form

A finite difference form of the governing equation was created by applying the proper 3 point 2nd derivatives in the time and space directions [1]. The finite difference form is given below. Notice that y_P^+ is the y at the advanced time and y_P^- is the y at the old time.

$$\frac{y_E - 2y_P + y_W}{\Delta x^2} = \frac{1}{c^2} \frac{y_P^+ - 2y_P + y_P^-}{\Delta t^2}$$

When rearranged, the equation is non-dimensionalized by the problem parameters Δx, c and Δt. The rearranged form is given below.

$$y_E - 2y_P + y_W = \frac{\Delta x^2}{c^2 \Delta t^2}\left(y_P^+ - 2y_P + y_P^-\right)$$

When $\Delta x^2 / c^2 \Delta t^2$ is set to unity, the expression reduces to a simple time-step-limited form. This limiting case will be the general hyperbolic update equation. The equation is given below.

$$y_P^+ = y_E + y_W - y_P^-$$

The update equation only requires the neighboring values at the current time and one old time value.

For the initial step, an old time value is not known. Thus, the starting expression below is used to generate the first time advance when the solution starts from rest [1].

$$y_P^+ = \tfrac{1}{2}\left(y_E + y_W\right)$$

Solution Method

A computer program was written in Visual Basic to solve the hyperbolic wave equation for one dimension. The program is listed in the Appendix. The general program steps are given below.

1. **Problem Initialization** The program initializes the variables.

2. **Read Starting Values** Initial Values are read-in.

3. **Advance First Iteration** The program advances with the special *starting* equation.

4. **Time Loop** The time advance loop is entered

5. **Update Equation** The general update equation sweeps over the interior points

6. **Advance Solution** The solution values are updated for the next time step.

7. **Write Out** The current solution values are output to the spreadsheet.

Results

This project investigates the vibration and forced response of a long-wire antenna at the author's home. The antenna is 14 gauge copper coated steel wire that is 150 feet long. The wire is strung between a sturdy pole and a flexible tower. The antenna wire is tensioned to 50 pounds. The wire weighs 0.012 pounds per linear foot. This project will investigate the wave speed and wave shape of the antenna wire.

Wave Speed

The first objective is to find the wave speed. From theory, the wave speed is $\sqrt{T/\rho}$ or approximately 260 feet per second. With a total traveling length of 300 feet, a wave will take slightly over 1 second to complete a *trip*.

From previous experimentation, it was estimated that a wave required approximately 1 second to return to the sender. The wave speed approximately matches experimental values.

The computations predict a wave traveling 150 feet in 0.5769 seconds. This is 260 feet per second.

Wave Shape

Finally, the several wave distributions were determined. Shapes were found for the fundamental frequency, the first harmonic, a transient case and a pulse function.

Fundamental Frequency

Next, the string's fundamental frequency is determined. Determining the frequency from experiment is trivial. The frequency is the inverse of period or approximately 1 Hz.

From the harmonic solution given above, the frequency is,

$$f = n \cdot \frac{c}{L} = \frac{260}{300} = 0.867 \, \text{Hz}$$

For the computational test, the program started with one half of a sine wave. Figure 1 shows the computed displacement contours.
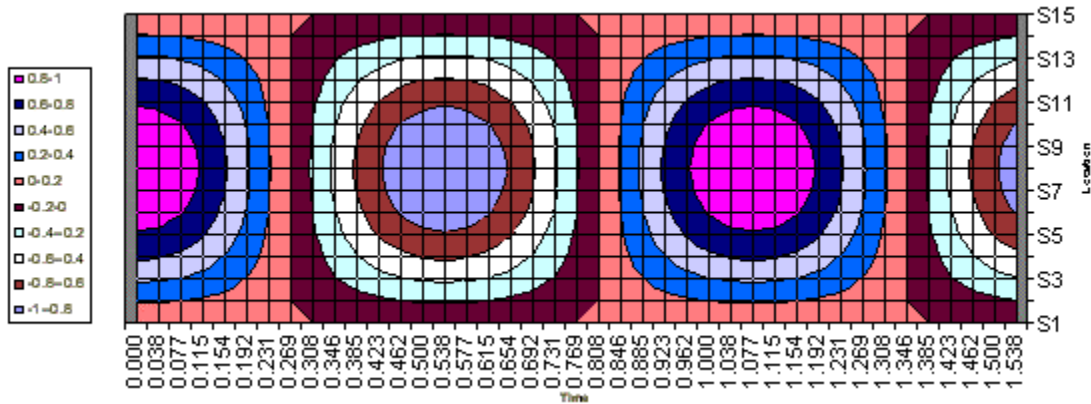

Figure 1. Fundamental

Clearly, the solution is the fundamental frequency. Each cycle requires 1.15 seconds. This is a frequency of 0.867 Hz. The computer code is working properly.

First Harmonic

This test determines the wave shapes for the first harmonic (twice fundamental frequency). Figure 2 shows the wave pattern for the first harmonic.
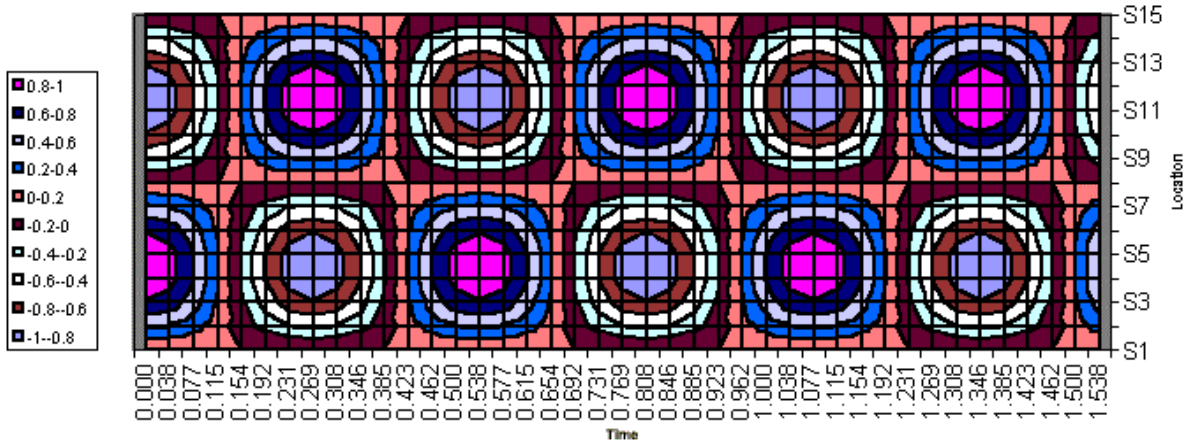

Figure 2. First Harmonic

Notice that increasing the temporal frequency increases the spatial frequency. The solution starts and continues with all of the starting transients damped out. This test approximates the antenna wire vibrating with stationary end points.

Transient First Harmonic

This test determines the wave response to the starting transients. A sinusoidal forcing function on the bottom creates a wave pattern response. The forcing function frequency is exactly that of the string's first harmonic. Figure 3 shows the transient startup case. Time advances from left to right.
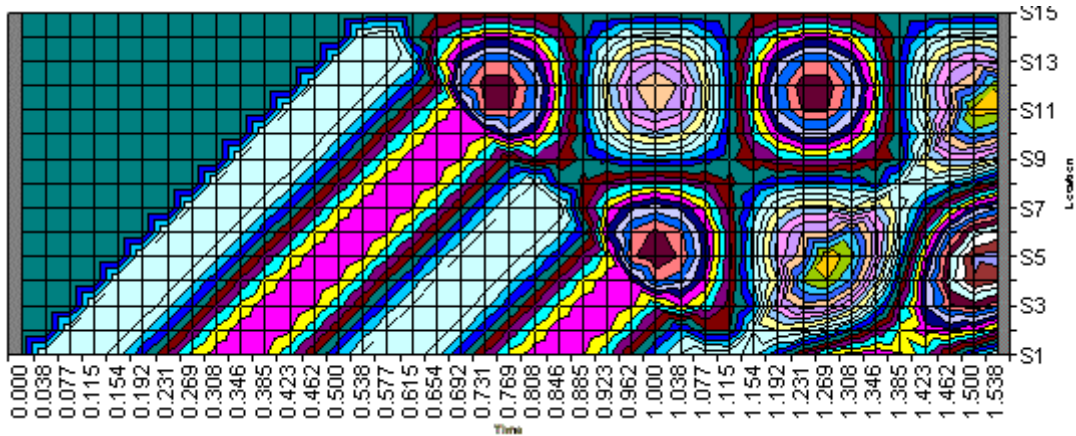

Figure 3. Transient

This test approximates the antenna vibration caused by the oscillating tower. The forcing function appears as a traveling wave until the top reflection causes standing waves to be visible. Notice that the standing waves are complete by one reflection time. The traveling waves are still present, however they combine to form standing waves.

Pulse Wave

Because the governing equation allows any traveling wave as a solution, the forcing function can incorporate *any* waveform or frequency. Figure 4 shows the response to two pulse waves.
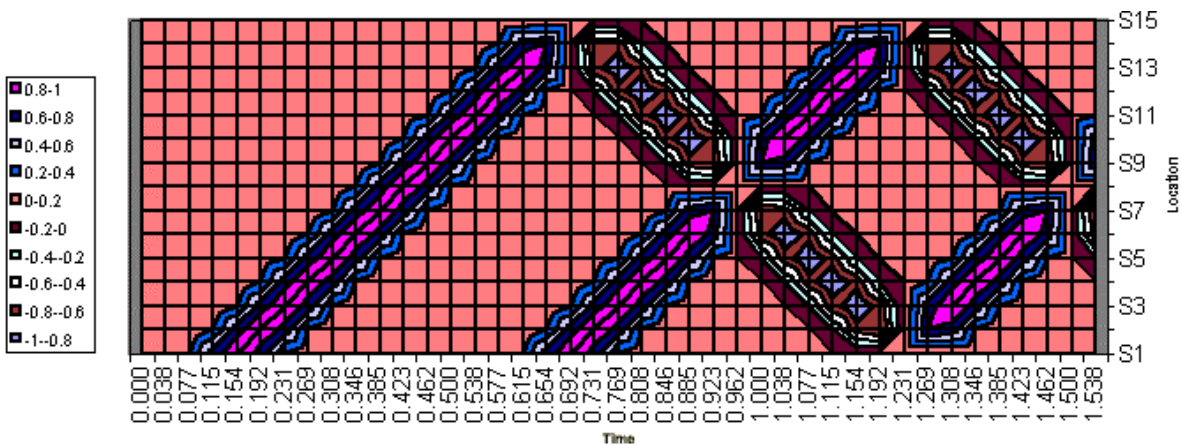

Figure 4. Pulse Wave

Notice that the reflected wave has an inverted amplitude. This is due to the no-movement boundary condition at the top. Partial destructive waves can occur where the wave fronts meet. This case approximates the *human* generated waves in the antenna system.

This test shows how the computer model corresponds to reality. In reality, a single pulse will become diffused and less distinct as the wave progresses along the wire. After a few cycles, a

single pulse will start degenerating into several harmonics. This is probably due to two processes: boundary conditions and wire stiffness. The first process occurs because the boundary conditions are not completely fixed on both ends. There is some damping associated with the pulse reflecting off the end points. The second process occurs because the wire has stiffness, which makes the wire act as a beam.

Conclusions

A finite difference 1D string vibration solver was successfully implemented. A Visual Basic program was used in Excel for the computations. Four testcases were used to show the wave properties and the wave shapes of forced and free vibration.

The results show that the finite difference wave equation solver works surprisingly well. The frequencies and wave shapes match those of an exact solution. A transient testcase visually presented the transformation of a two traveling waves into a single standing wave. A pulse function shows that arbitrary traveling waves satisfy the wave equation. These computations only consider perfect boundary conditions, which do not occur in reality.

The wave equation solver performed surprising well. Frequencies and wave shapes for steady state and transient vibrations were easily programmed and solved. The resulting solutions were theoretically correct.

References

[1] Lilley, D. G., *Numerical Methods*, Stillwater, OK, 2002.

[2] Kinsler, L.E, et al, *Fundamentals of Acoutics*, Wiley & Sons, 2000.

Appendix: Computer Code

```
Option Explicit

Sub main()
' Project 3
' MAE 5093
' Charles O'Neill #8
' 7 June 2002

'Initilization
Dim Y(100) As Double, YNEW(100) As Double, YOLD(100) As Double
Dim i As Integer, imax As Integer, imm1 As Integer
Dim iter As Integer, itermax As Integer
Dim t As Double, delx As Double, delt As Double, c As Double

'
c = 260
imax = 15
imm1 = imax - 1
itermax = 200
delx = 10
delt = delx / c

'Read in Starting Values
  For i = 1 To imax
    YOLD(i) = ActiveSheet.Cells(6, 5 + (i - 1))
  Next i

'Advance First Time (Special Advance) ------------------
  For i = 2 To imm1
    Y(i) = 0.5 * (YOLD(i + 1) + YOLD(i - 1))
  Next i
' Write Out Values
  ActiveSheet.Cells(7, 4) = delt
  For i = 1 To imax
    ActiveSheet.Cells(7, 5 + (i - 1)) = Y(i)
  Next i


'Advance Time --------------------------------------------
For iter = 2 To itermax

' Sweep Field
  For i = 2 To imm1
'   Update Equation
    YNEW(i) = Y(i + 1) + Y(i - 1) - YOLD(i)
  Next i

' Advance Solution Values
  For i = 2 To imm1
    YOLD(i) = Y(i)
    Y(i) = YNEW(i)
  Next i

' Write Out Values
  ActiveSheet.Cells(6 + iter, 4) = delt * iter
  For i = 1 To imax
    ActiveSheet.Cells(6 + iter, 5 + (i - 1)) = Y(i)
  Next i

Next iter
End Sub
```