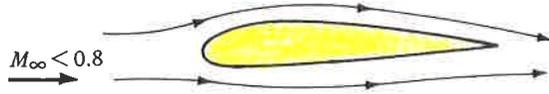


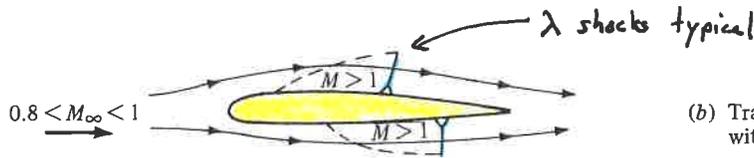
Lesson 5  
Fluid Mechanics  
+  
Conservation Laws

# Flow Regimes

AEM 313  
+  
AEM 614

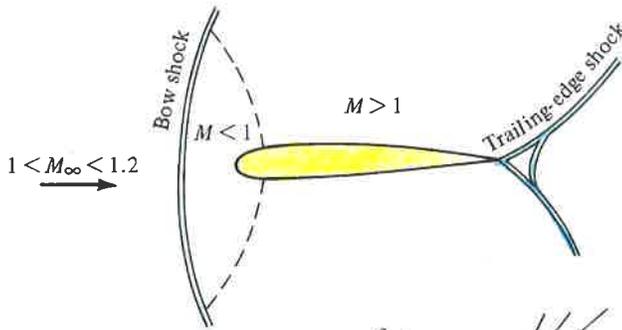


(a) Subsonic flow



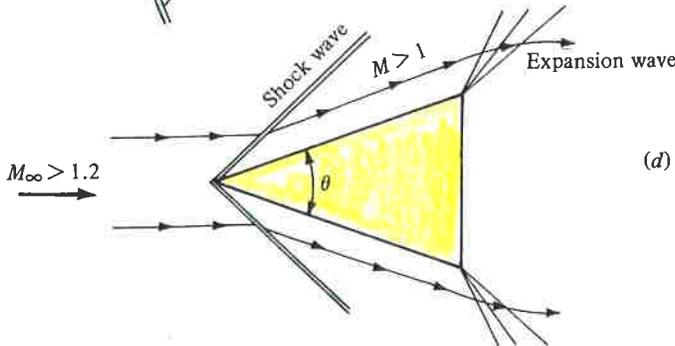
(b) Transonic flow with  $M_\infty < 1$

AEM 614  
+  
CFD  
+  
Wind tunnel



(c) Transonic flow with  $M_\infty > 1$

AEM 413



(d) Supersonic flow

AEM 624



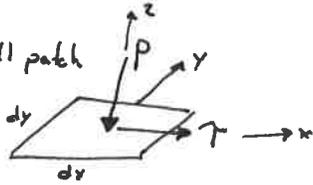
(e) Hypersonic flow

Figure 1.44 Different regimes of flow.

Source: F.O.A., Anderson

# Pressure and Shear Stresses generate Forces and Moments

- On a small patch

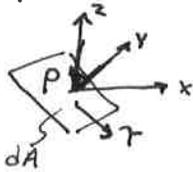


$$F_z = P \cdot \text{Area} = p \, dx \, dy$$

$$F_x = \tau \cdot \text{Area} = \tau \, dx \, dy$$

Unit:  $\frac{\text{lb}_f}{\text{in}^2} \cdot \frac{\text{in}}{1} \cdot \frac{\text{in}}{1} = \text{lb}_f \text{V}$

- On a small patch at an angle



$p$  acts in normal direction



$$\hat{n} = (\sin \theta, \cos \theta, 0)$$

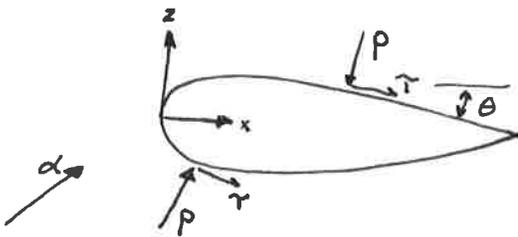
$$\vec{F} = (P)(\hat{n})(\text{Area}) = p \hat{n} \, dA$$

$$= p \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} dA$$

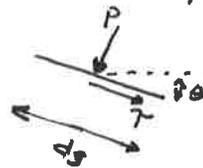
ways to understand/visualize

- pressure is identical in any frame, so the normals apply to  $dA$
- pressure decomposed to  $\hat{n}$  and applied to area.

- 2D airfoil (units span)



- On the upper surface, the body frame Normal force is:



$$dN'_u = \underbrace{-P_u \cos \theta \, ds}_{\text{component in body z frame}} \underbrace{- \tau_u \sin \theta \, ds}_{\substack{+ \text{ pressure gives} \\ \text{down force}}}$$

+ shear gives down force

The Axial component is

$$dA'_u = -P_u \sin \theta \, ds + \tau_u \cos \theta \, ds$$

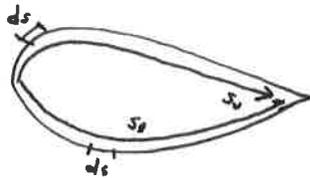
- On the lower surface, the  $\left\{ \begin{array}{l} \text{Normal force} \\ \text{Axial force} \end{array} \right.$  is:



$$dN'_l = P_l \cos \theta \, ds - \tau_l \sin \theta \, ds$$

$$dA'_l = P_l \sin \theta \, ds + \tau_l \cos \theta \, ds$$

Integrating the pressure and shear gives Forces and Moments



Split into two integral parts: Upper + Lower  
 $ds$  is a small section along  $s$

Ex: Arc Length =  $\int_{LE}^{TE} ds_u + \int_{TE}^{LE} ds_l = \text{circled arrow}$

Total Normal and Axial Force

Integrate  $dN_u$   $dN_l$   $dA_u$   $dA_l$  over  $s$ .

$$N' = \underbrace{\int_{LE}^{TE} dN_u'}_{\text{upper}} + \underbrace{\int_{LE}^{TE} dN_l'}_{\text{lower}} = - \int_{LE}^{TE} (P_u \cos \theta + \tau_u \sin \theta) ds_u + \int_{LE}^{TE} (P_l \cos \theta - \tau_l \sin \theta) ds_l$$

$$A' \text{ (similar)} = \int_{LE}^{TE} (-P_u \sin \theta + \tau_u \cos \theta) ds_u + \int_{LE}^{TE} (P_l \sin \theta + \tau_l \cos \theta) ds_l$$

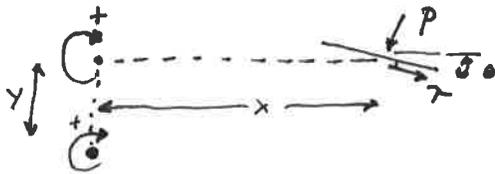
In global frame

$$C_R' = C_N \cos \alpha - C_A \sin \alpha$$

$$C_d' = C_N \sin \alpha + C_A \cos \alpha$$

Moment

The Normal and Axial forces acting at a distance from a ref pt.



$$dM' = \underbrace{-dN \cdot x}_{\substack{+ x \sin \alpha \\ + M}} + \underbrace{dA \cdot y}_{\substack{+ y \cos \alpha \\ + M}}$$

Similar to above

$$dM_{LE} = \int_{LE}^{TE} (P_u \cos \theta + \tau_u \sin \theta) x ds_u - (P_u \sin \theta - \tau_u \cos \theta) y ds_u + \int_{LE}^{TE} (-P_l \cos \theta + \tau_l \sin \theta) x + (P_l \sin \theta + \tau_l \cos \theta) y ds_l$$

Dimensionless

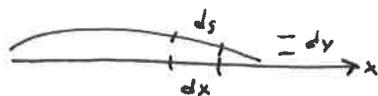
$$C_N = \frac{N}{\rho S}$$

$$C_M = \frac{M}{\rho S c}$$

$$C'_N = \frac{N'}{\rho c}$$

$$C'_M = \frac{M'}{\rho c^2}$$

Geometry



$$dx = \cos \theta ds$$

$$dy = -\sin \theta ds$$

$$\Rightarrow ds = \frac{dx}{\cos \theta}$$

$$ds = \frac{-dy}{\sin \theta}$$

N from above, divide by  $\rho c$

$$C'_N = \frac{1}{\rho c} \int_{LE}^{TE} P d... = \frac{1}{c} \int_0^c \frac{P}{\rho} \dots = \frac{1}{c} \int_0^c C_p \dots$$

$$= -\frac{1}{\rho c} \int_0^c \left( \frac{P_u}{\rho} \cos \theta + \frac{\tau_u}{\rho} \sin \theta \right) ds + \frac{1}{c} \int_0^c \left( \frac{P_r}{\rho} \cos \theta - \frac{\tau_r}{\rho} \sin \theta \right) ds$$

for these terms use  $ds = -\frac{dy}{\sin \theta}$

for these terms, use

$$ds = \frac{dx}{\cos \theta}$$

$$= + \frac{1}{c} \int_0^c \left[ C_{p_u} (-dx) + C_{f_u} (+dy) + C_{p_r} (dx) + C_{f_r} (dy) \right]$$

and  $dy = \frac{dy}{dx} dx$

$$C'_N = \frac{1}{c} \int_0^c (C_{p_r} - C_{p_u}) dx + \frac{1}{c} \int_0^c \left( C_{f_u} \frac{dy_u}{dx} + C_{f_r} \frac{dy_r}{dx} \right) dx$$

difference  
in pressure

pressure component

shape  
of  
surface

skin friction  
component

Similar process for  $C_a'$  and  $C_m'$

$$C_a' = \underbrace{\frac{1}{c} \int_0^c \left( C_{p_u} \frac{dy_u}{dx} - C_{p_e} \frac{dy_e}{dx} \right) dx}_{\text{pressure}} + \underbrace{\frac{1}{c} \int_0^c (C_{f_u} + C_{f_e}) dx}_{\text{skin friction}}$$

$$C_m' = \frac{1}{c^2} \int_0^c (C_{p_u} - C_{p_e}) x dx - \frac{1}{c^2} \int_0^c \left( C_{f_u} \frac{dy_u}{dx} + C_{f_e} \frac{dy_e}{dx} \right) x dx$$
$$+ \frac{1}{c^2} \int_0^c \left( C_{p_u} \frac{dy_u}{dx} + C_{f_u} \right) y_u dx + \frac{1}{c^2} \int_0^c \left( -C_{p_e} \frac{dy_e}{dx} + C_{f_e} \right) y_e dx$$

and

$$C_l = C_n \cos \alpha - C_a \sin \alpha$$

$$C_d = C_n \sin \alpha + C_a \cos \alpha$$

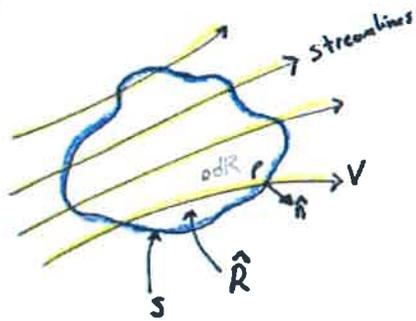
- Does this mean that we need the pressure and viscous stress across the entire wing/aircraft/airfoil in order to calculate lift and drag?

Thank goodness NO

- How is shear stress at the wall measured?

Not easy!

• Continuity of Mass (2D)



$$\text{Total mass} = \int_R \rho dR$$

Along the surface, the flux (mass per unit time) is  $F = \rho V \cdot n dS$

English:

A positive rate of mass leaving the control volume leads to a proportional rate of decrease in the mass inside the CV.

Math:

$$\int_S F dS = -\frac{d}{dt} \int_R \rho dR \Rightarrow \int_S \rho V \cdot \hat{n} dS = -\frac{d}{dt} \int_R \rho dR$$

Green's Theorem (Divergence Theorem)

$$\int_S \tilde{V} \cdot \hat{n} dS = \int_R \text{div } \tilde{V} dR$$

where  $\tilde{V}$  is  $\rho V$  for us

substitute into continuity of mass to replace surface integral with volume integral

$$\int_R \text{div } \rho V dR = -\frac{d}{dt} \int_R \rho dR = -\int_R \frac{d\rho}{dt} dR$$

Combine + Simplify

$$\int_R \left[ (\text{div } \rho V) + \left( \frac{d\rho}{dt} \right) \right] dR = 0$$

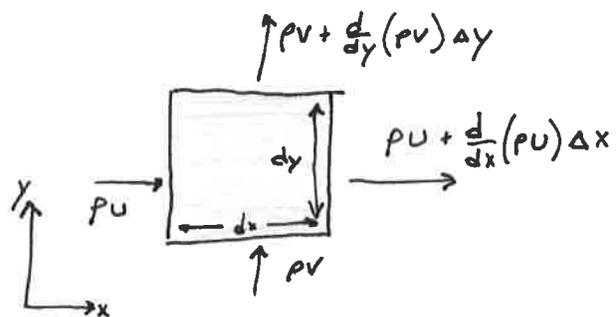
Since R is arbitrary

$$\boxed{\frac{d\rho}{dt} + \text{div } \rho V = 0}$$

Mass Continuity:

The change in density at a point is equal to the negative divergence of  $\rho V$

Alternative Control Volume approach to finding Mass Conservation Gov Egu.



• Look at mass flux through square Control Volume

• Taylor series for variables

$$\rho u \Big|_{\text{right}} = \rho u \Big|_{\text{left}} + \frac{d(\rho u)}{dx} \Delta x$$

$$\rho v \Big|_{\text{top}} = \rho v \Big|_{\text{bottom}} + \frac{d(\rho v)}{dy} \Delta y$$

• Track the total influx/outflux of mass :

Influx/Entering

$$\underbrace{(\rho u) \Delta y}_{\text{left}} - \underbrace{\left(\rho u + \frac{d}{dx}(\rho u) \Delta x\right) \Delta y}_{\text{right}} + \underbrace{(\rho v) \Delta x}_{\text{bottom}} - \underbrace{\left(\rho v + \frac{d}{dy}(\rho v) \Delta y\right) \Delta x}_{\text{top}}$$

Entering mass increases the total mass within the C.V.

$$\frac{d(\rho \Delta x \Delta y)}{dt}$$

Together:

$$\begin{aligned} \frac{d}{dt}(\rho \Delta x \Delta y) &= \cancel{(\rho u) \Delta y} - \cancel{(\rho u) \Delta y} - \frac{d}{dx}(\rho u) \Delta x \Delta y \\ &+ \cancel{(\rho v) \Delta x} - \cancel{(\rho v) \Delta x} - \frac{d}{dy}(\rho v) \Delta x \Delta y \end{aligned}$$

Simplify

$$\frac{d\rho}{dt} \Delta x \Delta y = - \frac{d}{dx}(\rho u) \Delta x \Delta y - \frac{d}{dy}(\rho v) \Delta x \Delta y$$

Since  $\Delta x$  and  $\Delta y$  are arbitrary and non-zero, divide by  $\Delta x \Delta y$

$$\boxed{\frac{d\rho}{dt} + \frac{d}{dx}(\rho u) + \frac{d}{dy}(\rho v) = 0}$$

definition of divergence of  $\rho \vec{V}$

$$\boxed{\frac{d\rho}{dt} + \nabla \cdot (\rho \vec{V}) = 0}$$

For incompressible flows,  $\rho$  is constant.  $\frac{d\rho}{dt} = 0$

$$\boxed{\nabla \cdot (\rho \vec{V}) = 0}$$

is. Any gradient or change in  $\rho u$  in the  $x$  direction is matched by a change in  $\rho v$  in the  $y$  direction.

# Derive Momentum Gov Eqs.

Newton's law.

$$F = ma \Rightarrow \vec{F} = m \frac{d\vec{V}}{dt}$$



But the velocity of the fluid <sup>particle</sup> depends on location and time.

Chain rule!!

$$\frac{dV}{dt} = \underbrace{\frac{\partial V}{\partial x}}_x \frac{dx}{dt} + \underbrace{\frac{\partial V}{\partial y}}_y \frac{dy}{dt} + \underbrace{\frac{\partial V}{\partial z}}_z \frac{dz}{dt} + \underbrace{\frac{\partial V}{\partial t}}_t$$

aka:  $\frac{DV}{Dt}$

But what is  $\frac{dx}{dt}$ ? Change in x location with respect to time. U Velocity

$$\frac{dx}{dt} = U$$

So,

$$\underbrace{\frac{DV}{Dt}}_{\substack{\text{Notations for} \\ \text{"total"} \\ \text{or} \\ \text{"convective"} \\ \text{Substantial} \\ \text{derivative}}} = \underbrace{\frac{\partial V}{\partial x} U + \frac{\partial V}{\partial y} V + \frac{\partial V}{\partial z} W}_{\substack{\text{Convective} \\ \text{derivatives}}} + \underbrace{\frac{\partial V}{\partial t}}_{\text{local derivative}}$$

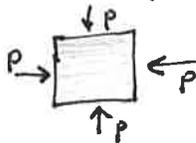
$$= (\vec{V} \cdot \nabla) \vec{V}$$

Total:

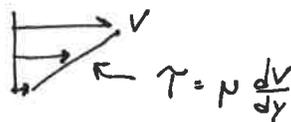
$$m \frac{DV}{Dt} = m (\vec{V} \cdot \nabla) \vec{V} + m \frac{dV}{dt} = \vec{F} \Rightarrow \rho \frac{dV}{dt} + \rho (\vec{V} \cdot \nabla) \vec{V} = \frac{\vec{F}}{Vol}$$

What are the forces on the fluid control volume?

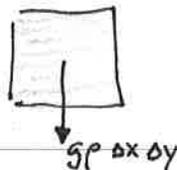
• pressure



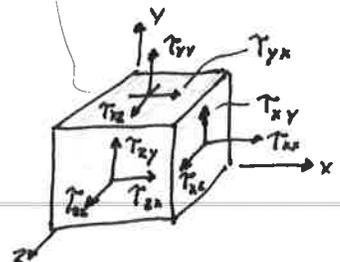
• Viscous



• Body

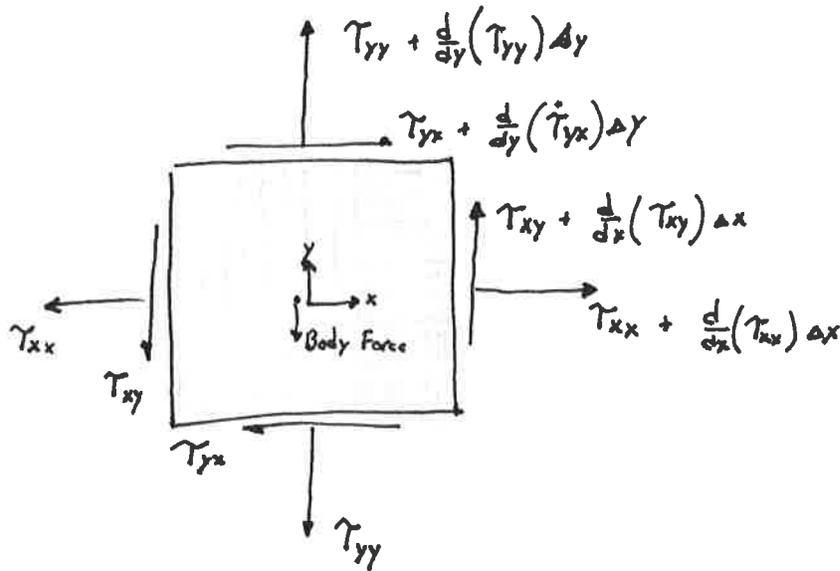


These are both stresses!



$\tau_{ij}$  = stress on "i" face in "j" direction.

## 2D Momentum Control Volume



Sum Forces:

$$\begin{aligned} \sum F_x &= \underbrace{\left[ -\tau_{xx} + \tau_{xx} + \frac{d}{dx}(\tau_{xx})\Delta x \right]}_{=0, \text{cancel}} \Delta y + \underbrace{\left[ -\tau_{yx} + \tau_{yx} + \frac{d}{dy}(\tau_{yx})\Delta y \right]}_{=0} \Delta x \\ &= \frac{d}{dx}(\tau_{xx}) \Delta x \Delta y + \frac{d}{dy}(\tau_{yx}) \Delta y \Delta x \end{aligned}$$

$\sum F_y = \text{similar}$

$$= \frac{d}{dy}(\tau_{yy}) \Delta x \Delta y + \frac{d}{dx}(\tau_{xy}) \Delta x \Delta y$$

Combine

$$m \frac{d\vec{V}_i}{dt} + m(\mathbf{V} \cdot \nabla) \vec{V}_i = \frac{d}{dx}(\tau_{xx}) \Delta x \Delta y + \frac{d}{dy}(\tau_{yx}) \Delta y \Delta x$$

Divide by  $\Delta x \Delta y$  and  $\Delta z = 1$

$$\rho \frac{dV_x}{dt} + \rho(\mathbf{V} \cdot \nabla) V_x = \frac{d}{dx}(\tau_{xx}) + \frac{d}{dy}(\tau_{yx})$$

Fluid model

$$\tau_{xx} = -p - \frac{2}{3} \mu \nabla \cdot \vec{V} + 2\mu \frac{du}{dx}$$

$$\tau_{xy} = \tau_{yx} = \mu \left( \frac{du}{dy} + \frac{dv}{dx} \right)$$

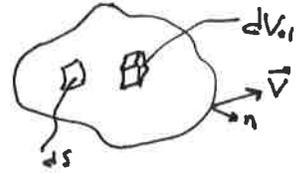
← this is only a model not a law for a Newtonian fluid.

OK for air but not for toothpaste.

# Integral Form of Conservation Laws.

Mass:

$$\underbrace{\iiint \frac{d\rho}{dt} dV_{ol}}_{\text{Integral over entire CV volume}} + \underbrace{\oint \rho \mathbf{V} \cdot \hat{\mathbf{n}} dS}_{\text{Surface Integral over CV}} = 0$$



"The time rate of change of density ~~must~~ plus the net outflow of mass ~~must~~ sum to zero"

Momentum

$$\underbrace{\iiint \frac{d}{dt} (\rho \vec{V}) dV_{ol}}_{\text{time rate of total momentum}} + \underbrace{\oint \rho (\vec{V} \cdot \hat{\mathbf{n}}) \vec{V} dS}_{\text{body force}} = \underbrace{\iiint \rho \mathbf{f} dV_{ol}}_{\text{body force}} + \underbrace{\oint \underbrace{-\rho \hat{\mathbf{n}}}_{\text{pressure}} dS}_{\text{pressure}} + \underbrace{\oint \underbrace{\vec{\tau} \cdot \hat{\mathbf{n}}}_{\text{viscous stresses}} dS}_{\text{viscous stresses}} + \underbrace{B}_{\text{Body force}}$$

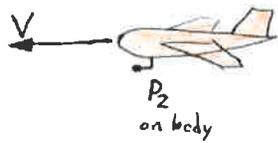
Energy

$$\underbrace{\iiint \frac{d}{dt} \left( \rho e + \frac{\rho V^2}{2} \right) dV_{ol}}_{\text{time rate of total energy in CV}} + \underbrace{\oint \rho \left( e + \frac{V^2}{2} \right) \vec{V} \cdot dS}_{\text{pressure work}} + \underbrace{\iiint \dot{q} \rho dV_{ol}}_{\text{heating}} + \underbrace{\dot{Q}_{\text{viscous}}}_{\text{viscous heating}}$$

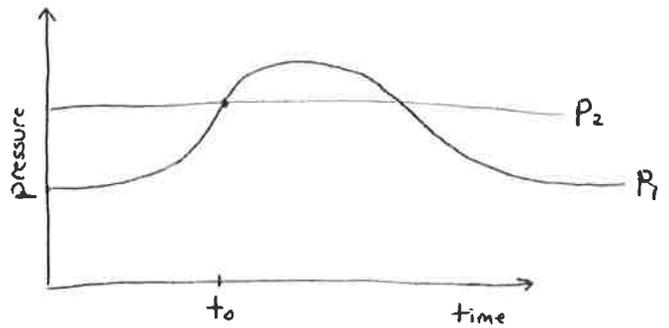
$$- \underbrace{\oint \rho \vec{V} \cdot dS}_{\text{pressure work}} + \underbrace{\iiint \rho (\mathbf{f} \cdot \mathbf{V}) dV}_{\text{body force work}} + \underbrace{\dot{W}_{\text{visc}}}_{\text{viscous work}}$$

# Derivative Reference Frames

$P_1$   
stationary



$$\frac{dp_1}{dt} \neq \frac{dp_2}{dt}$$



In general,  $p = p(x(t), y(t), z(t), t)$

$$\frac{dp}{dt} = \frac{dp}{dx} \frac{dx}{dt} + \frac{dp}{dy} \frac{dy}{dt} + \frac{dp}{dz} \frac{dz}{dt} + \frac{dp}{dt}$$

If following a particular fluid particle,  $\frac{dx}{dt} = u$     $\frac{dy}{dt} = v$     $\frac{dz}{dt} = w$

$$\frac{dp}{dt} = \underbrace{\frac{dp}{dx} u + \frac{dp}{dy} v + \frac{dp}{dz} w + \frac{dp}{dt}}_{\equiv \frac{Dp}{Dt}}$$

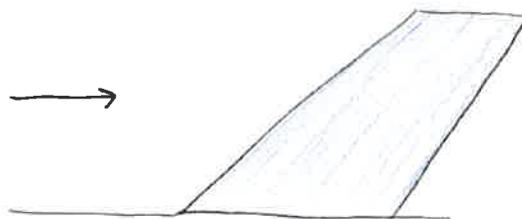
## Substantial Derivative

$$\underbrace{\frac{D(\quad)}{Dt}}_{\substack{\cdot \text{substantial} \\ \cdot \text{Material} \\ \cdot \text{Total} \\ \cdot \dots}} = \underbrace{\frac{d(\quad)}{dt}}_{\cdot \text{local}} + \underbrace{V \cdot \nabla(\quad)}_{\cdot \text{convective}}$$

$\frac{D}{Dt}$  is the derivative following a particle in the flow

$\frac{d}{dt}$  is the derivative at a particular location

Ex: Given a swept wing at zero AOA, describe the flow behavior near the LE. Assume inviscid, incompressible, steady



Eulerian frame:

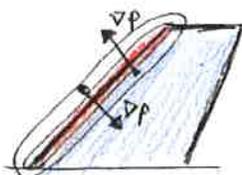
$$\frac{d(\rho V)}{dt} + \nabla \cdot (\rho V V T) = -\nabla p \quad \text{and} \quad \frac{d\rho}{dt} + \nabla \cdot (\rho V) = 0$$

Now what?

Lagrangian frame:

$$\rho \frac{DV}{Dt} = -\nabla p \quad \text{and} \quad \frac{D\rho}{Dt} + \rho \nabla \cdot V = 0$$

We know that a high pressure stagnation pressure forms along the LE



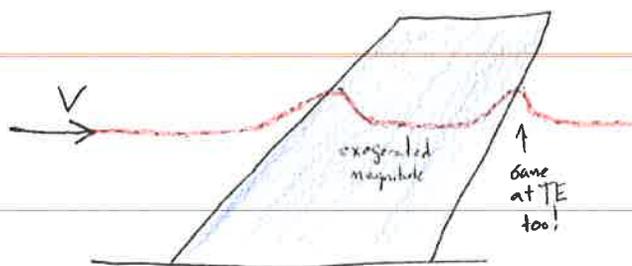
The pressure gradient  $\nabla p$  is not aligned with the flow!

The incoming flow 'sees'/feels a  $-y$  component in  $\nabla p$  ahead of the LE and a  $+y$  component of  $\nabla p$  aft of the LE.

$$\frac{DV_y}{Dt} = -\frac{\nabla p}{\rho} = -\frac{(-)}{\rho} = \frac{+}{\rho}$$

$$\frac{DV_y}{Dt} = -\frac{\nabla p}{\rho} = -\frac{(+)}{\rho} = \frac{-}{\rho}$$

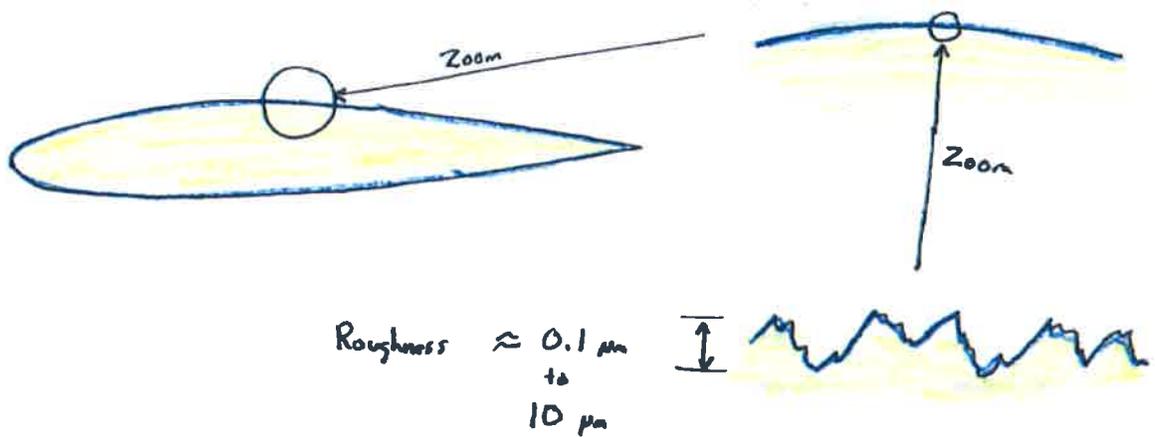
The particle deflects outboard in front of the LE and inboard aft of the LE



We actually see this in real wings.

The Lagrangian frame can show amazing insights.

# Boundary Conditions



Size of  $N_2$  is about  $100 \mu\text{m}$

$\lambda_{SSL} \approx 65 \text{nm}$

Ratio of sizes (roughness to  $N_2$  size)

$$1 \mu\text{m} : 100 \mu\text{m} \Rightarrow 1 \times 10^{-6} : 1 \times 10^{-12} \Rightarrow 1 : 1 \text{ million}$$

Ratio of sizes (roughness to mean free path)

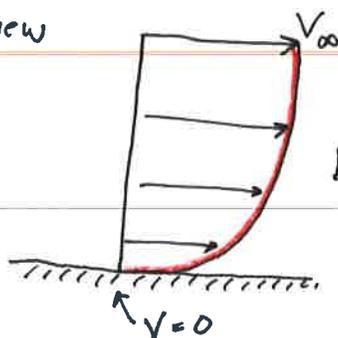
$$65 \text{nm} : 100 \mu\text{m} \Rightarrow 1 : 650$$

X direction momentum is constrained/limited within the roughness height.

Viscous boundary conditions are essentially no-slip

$$V_b = 0$$

Macro View



Boundary Layer profile of Velocity  
(concept only!)

# Non-Dimensionalization

$$\rho \frac{dh_0}{dt} + \rho V \cdot \nabla h_0 = \frac{dp}{dt} + \nabla \cdot \left( \bar{T} \cdot V \right) - \nabla \cdot \dot{q}$$

$\rho \frac{du}{dx}$        $k \frac{dT}{dx}$

Pick dimensional terms + reference values (EVA p 10)

Substitute

$$t = t^* \frac{L_r}{V_r} \quad \text{etc.} \quad V = V^* V_r \quad h = h^* a_r^2 \quad p = p^* p_r \quad \rho = \rho^* \rho_r V_r^2$$

$$\rho^* \rho_r \frac{d(h^* a_r^2)}{d(t^* \frac{L_r}{V_r})} + \rho^* \rho_r V^* V_r \cdot \frac{\nabla^* h^* a_r^2}{L_r} = \frac{d(\rho^* p_r)}{d(t^* \frac{L_r}{V_r})} + \frac{\nabla^*}{L_r} \cdot \left( \mu_r \frac{V_r}{L_r} \bar{T}^* \cdot V^* V_r \right)$$

pull all ref values together.

$$\left( \frac{\rho_r a_r^2 V_r}{L_r} \right) \rho^* \frac{dh^*}{dt^*} + \left( \frac{\rho_r V_r a_r^2}{L_r} \right) \rho^* V^* \cdot \nabla^* h^* = \left( \frac{p_r V_r}{L_r} \right) \frac{dp^*}{dt^*} + \left( \frac{\mu_r V_r V_r}{L_r L_r} \right) \nabla^* \cdot (\bar{T}^* \cdot V^*) - \left( \frac{k_r a_r^2}{L_r^2 C_p} \right) \nabla^* \cdot \dot{q}^*$$

Divide by  $\frac{\rho_r a_r^2 V_r}{L_r}$

$$\rho^* \frac{dh^*}{dt^*} + \rho^* V^* \cdot \nabla^* h^* = \underbrace{\left( \frac{p_r V_r}{L_r} \right) \left( \frac{L_r}{\rho_r V_r a_r^2} \right)}_{V_r^2 / a_r^2 = M_{ref}^2} \frac{dp^*}{dt^*} + \underbrace{\left( \frac{\mu_r V_r^2}{L_r^2} \right) \left( \frac{L_r}{\rho_r V_r a_r^2} \right)}_{\frac{\mu_r}{\rho_r V_r L_r} = \frac{1}{Re_{ref}}} \nabla^* \cdot (\bar{T}^* \cdot V^*) - \underbrace{\left( \frac{k_r a_r^2}{L_r^2 C_p} \right) \left( \frac{L_r}{\rho_r V_r a_r^2} \right)}_{\frac{\nu}{L_r \rho_r V_r}} \nabla^* \cdot \dot{q}^*$$

$$\rho^* \frac{dh^*}{dt^*} + \rho^* V^* \cdot \nabla^* h^* = M_{ref}^2 \frac{dp^*}{dt^*} + \frac{M_{ref}^2}{Re_{ref}} \nabla^* \cdot (\bar{T}^* \cdot V^*) - \frac{1}{Re_{ref}} \nabla^* \cdot \dot{q}^*$$

- Fluid flows depend on
- Mach #
  - Reynolds #
  - Prandtl #
  - Equation of state