

Lesson 6

Integral Lift and Drag
+
Power Balance

Ex: Substantial Derivative

- On a standard day, an aircraft is climbing at 500 ft/min at an altitude of 10000 ft . What is the time rate of change of the temperature probe?

$$\lambda_{std} = -0.003566 \frac{R}{ft} \quad \text{lapse rate below } 36000 \text{ ft for Std-day}$$

We want the "particle" derivative on the aircraft (i.e. Total derivative)

$$\frac{DT}{Dt} = \underbrace{\frac{\partial(T)}{\partial t}}_{\substack{\text{rate of change} \\ \text{on aircraft}}} + \underbrace{V \cdot \nabla(T)}_{\substack{\text{rate of change} \\ \text{due to aircraft} \\ \text{flying through spatially} \\ \text{varying atmosphere}}}$$

$\frac{D}{Dt}$

Latin "notate bene" "take notice"

N.b. $\frac{dT}{dt} =$ Temperature rate of change at a fixed point

$\frac{DT}{Dt} =$ Temperature rate of change at a moving point

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \cdot \begin{pmatrix} \frac{dT}{dx} \\ \frac{dT}{dy} \\ \frac{dT}{dz} \end{pmatrix}$$

$$= \frac{\partial T}{\partial t} + \begin{pmatrix} 0 \\ 0 \\ 500 \text{ ft/min} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -0.003566 \frac{R}{ft} \end{pmatrix}$$

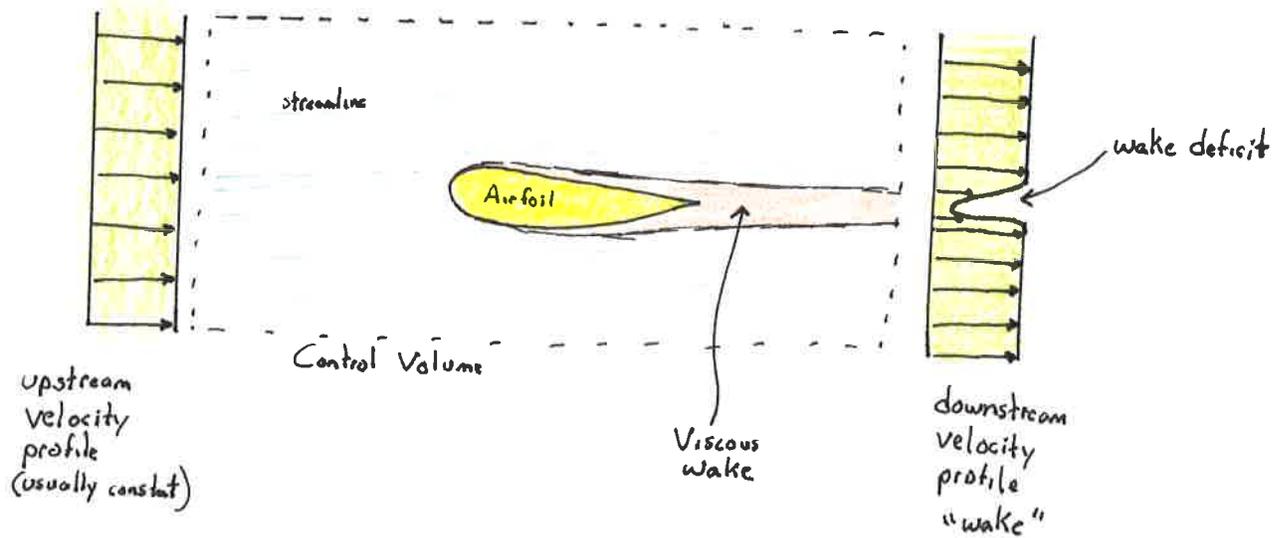
$$\boxed{\frac{DT}{Dt} = -1.78 \frac{R}{min}}$$

- Now, the atmosphere is cooling at $3 \frac{R}{min}$ uniformly at all altitudes.

$$\frac{\partial T}{\partial t} = -3 \frac{R}{min}$$

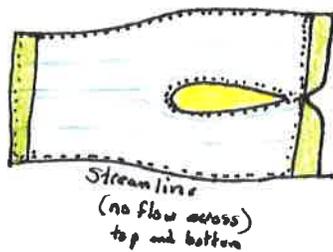
$$\frac{DT}{Dt} = \underbrace{\frac{\partial T}{\partial t}}_{-3 \frac{R}{min}} + \underbrace{V \cdot \nabla(T)}_{-1.78 \frac{R}{min}}$$

$$\boxed{\frac{DT}{Dt} = -4.78 \frac{R}{min}}$$

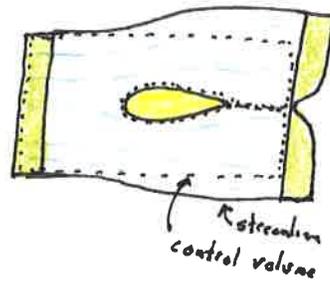


Important points and observations

- The upstream velocity profile is usually constant
- The airfoil creates a boundary layer from the no-slip condition. Away from the airfoil, the BL is called a wake.
- The downstream velocity profile includes a velocity deficit from the BL.
The loss of momentum in the wake directly corresponds to the drag force
- The choice of control volume impacts which terms are constant and which vary. Observe that fluid may exit the top and bottom due to the flow field developed by the airfoil. (i.e. the control volume may or may not follow streamlines)



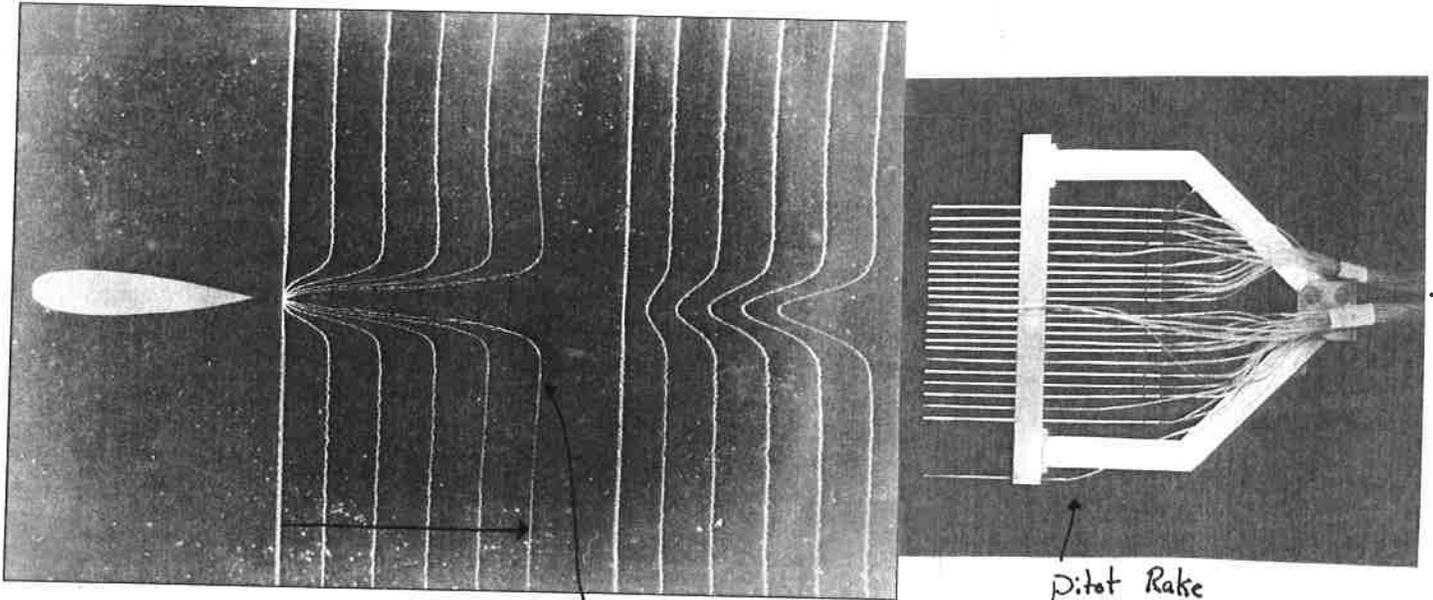
VS



please watch "Boundary Layer Control" at web.mit.edu/hml/ncfmf.html

Application of the Momentum Equation in Integral Form to determine Lift and Drag

$$\frac{d}{dt} \iiint_{V_i} \rho \vec{V} dV_i + \iint_S (\rho \vec{V} \cdot d\vec{S}) \vec{V} = - \iint_S p d\vec{S} + \iiint_V \rho f dV_i + F_{\text{vis}}$$



Flow direction →

↑ Flow visualization technique for water tunnels. Generate bubbles with electrical wire.

At TE, velocity is zero.

↑ The bubbles convect downstream

↑ Another wire generates bubbles

↑ Notice that this downstream wake has a non-zero velocity at the centerline "viscosity fills in velocity."

Later, we will learn why this velocity appears slightly faster than V_∞ !!

Pitot Rake measures flow velocities.

Notice the spacing increases away from the center.

↓ To manometer or pressure transducer.

Derivation of drag equation for a control volume approach.

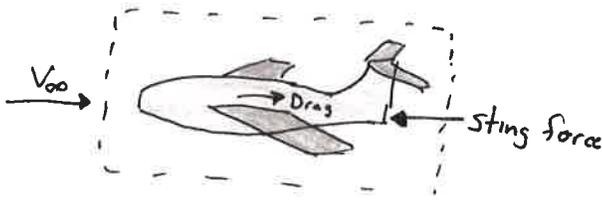
- Momentum Gov Equ.

$$\underbrace{\iiint_V \frac{d}{dt}(\rho \vec{V}) dV}_{\text{Steady state}} + \iint_S \rho(\vec{V} \cdot \hat{n}) \vec{V} dS = \underbrace{\iiint_V \rho \vec{f} dV}_{\substack{\text{distributed} \\ \text{No body force}}} + \iint_S -p \hat{n} dS + \iint_S \vec{\tau} \cdot \hat{n} dS + B$$

Careful C.V

- Drag

To maintain the model in the wind tunnel, a force must be constantly applied to the model



- This force goes through the control volume.
 - Or consider a body force B applied to the model such that the C.V is in equilibrium.
- $$B_x = -\text{Drag}$$

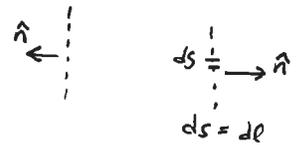
- Simplify Equ

$$\iint_S \rho(\vec{V} \cdot \hat{n}) \vec{V} dS = -\iint_S p \hat{n} \cdot dS + \vec{B} \begin{pmatrix} -\text{Drag} \\ -\text{Sideforce} \\ -\text{Lift} \end{pmatrix}$$

- Consider only the drag (x component) on the left and right faces

$$\text{Drag} = -\iint_S \rho(\vec{V} \cdot \hat{n}) \vec{V}_x dS - \iint_S p \hat{n}_x dS$$

$$= -\iint_S \rho \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} V_x dS - \iint_S p \hat{n}_x dS$$



- Upstream face (left)

$$\hat{n} \leftarrow \quad \hat{n} = (-1, 0, 0)^T$$

- Downstream face (right)

$$\rightarrow \hat{n} \quad \hat{n} = (1, 0, 0)^T$$

- Constant pressure Integral

Q: What is $\iint_S p_{\infty} \hat{n} dS$?

A: 0

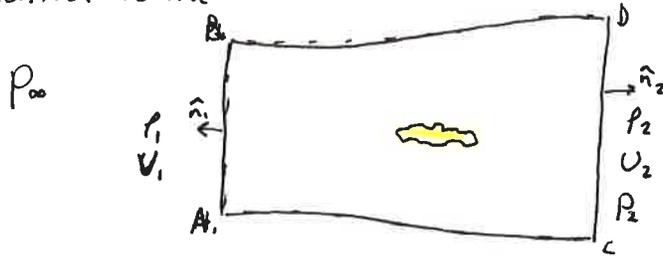


So, we can add an arbitrary pressure offset to the pressure term.

• Drag

$$D_{\text{drag}} = - \iint_S \rho V_x n_x V_x d\ell + \iint_S (P_{\infty} - p) n_x d\ell$$

• Control Volume



• Drag

$$D_{\text{drag}} = - \iint_S p_1 u_1 \hat{n}_x^1 u_1 d\ell - \iint_S p_2 u_2 \hat{n}_x^2 u_2 d\ell + \iint_S (P_{\infty} - p_1) \hat{n}_x^1 d\ell + \iint_S (P_{\infty} - p_2) \hat{n}_x^2 d\ell$$

• Mass Continuity

Steady

$$\iiint_V \frac{d}{dt} (\rho) dV + \iint_S \rho \mathbf{V} \cdot \hat{\mathbf{n}} dS = 0$$

in x dir only and applied to C.V. $\Rightarrow \iint_S -p_1 \mathbf{V}_x^1 d\ell + \iint_S p_2 \mathbf{V}_x^2 d\ell = 0$

Multiply by u_1 (a constant!)

$$\iint_A -p_1 u_1 u_1 d\ell + \iint_C p_2 u_1 u_2 d\ell = 0$$

Familiar term from mom' equ!!

• Drag

$$D_{\text{drag}} = \underbrace{\iint_C p_2 u_1 u_2 d\ell}_{\text{Combine to give}} - \iint_A p_2 u_2 u_2 d\ell - \iint_S (P_{\infty} - p_1) d\ell_1 + \iint_S (P_{\infty} - p_2) d\ell_2$$

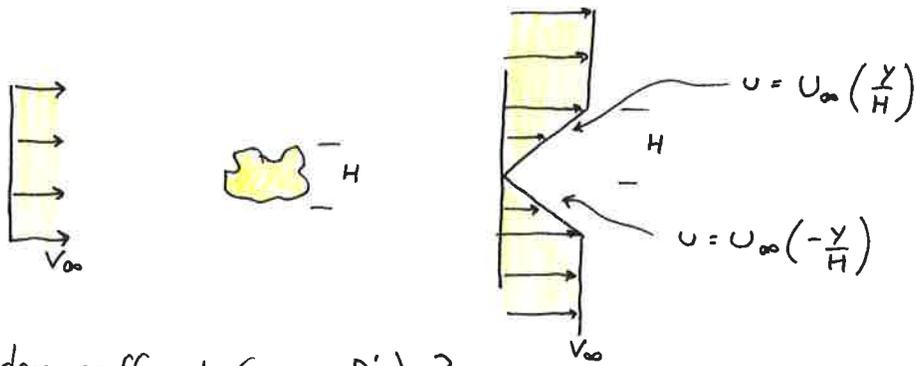
$P_1 = P_{\infty}$ Far enough upstream
 $P_2 = P_{\infty}$ Far enough downstream

$$D = \iint_C \underbrace{p_2 u_2}_{\text{Mass flux}} \underbrace{(u_1 - u_2)}_{\text{This is a velocity deficit}} d\ell$$

Generic result for a C.V away from shape being tested.

Near the shape/vehicle, you must include the pressure terms.

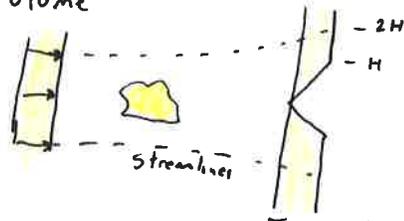
Ex:



Q: What is the drag coefficient ($C_d = \frac{D'}{\rho H}$)?

A:

• Control Volume



This fits the assumptions used in our derivation above.

← verify far enough from shape.

• Drag

$$\begin{aligned}
 D &= \int_{-2H}^{2H} \int_{-\infty}^{\infty} \rho_2 U_2 (U_1 - U_2) \, dA \rightarrow dy && \text{split into 4 parts} \\
 &= \int_{-2H}^{-H} \rho_2 V_\infty (V_\infty - V_\infty) \, dy + \int_{-H}^0 \rho_2 U_\infty \left(-\frac{y}{H}\right) (V_\infty - U_\infty \left(-\frac{y}{H}\right)) \, dy + \int_0^H \rho_2 U_\infty \frac{y}{H} (V_\infty - U_\infty \frac{y}{H}) \, dy + \int_H^{2H} \rho_2 V_\infty (V_\infty - V_\infty) \, dy \\
 &= \rho_2 U_\infty^2 \int_{-H}^0 \left(-\frac{y}{H}\right) \left(1 + \frac{y}{H}\right) \, dy + \rho_2 U_\infty^2 \int_0^H \frac{y}{H} \left(1 - \frac{y}{H}\right) \, dy \\
 &= \rho_2 U_\infty^2 \left. \left(-\frac{y^2}{2H} - \frac{y^3}{3H^2} \right) \right|_{-H}^0 + \rho_2 U_\infty^2 \left. \left(\frac{y^2}{2H} - \frac{y^3}{3H^2} \right) \right|_0^H \\
 &= \rho_2 U_\infty^2 \left(\frac{H^2}{2H} + \frac{-H^3}{3H^2} \right) + \rho_2 U_\infty^2 \left(\frac{H^2}{2H} - \frac{H^3}{3H^2} \right) \\
 &= \rho_2 U_\infty^2 H \left(\frac{1}{2} - \frac{1}{3} \right) + \rho_2 U_\infty^2 H \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{1}{3} \rho_2 U_\infty^2 H
 \end{aligned}$$

No surprise that these have the same contribution!

$$C_d = \frac{D}{\rho H} = \frac{D}{\frac{1}{2} \rho U_\infty^2 H}$$

$$= \frac{\frac{1}{3} \rho_2 U_\infty^2 H}{\frac{1}{2} \rho U_\infty^2 H} = \frac{2}{3}$$

$$\boxed{C_d = \frac{2}{3}}$$

Lift Derivation from a control volume approach

Momentum Gov. Egu.

$$\iiint_V \frac{d}{dt} (\rho \vec{V}) dV + \iint_S \rho (\vec{V} \cdot \hat{n}) \vec{V} dS = \iint_S -p \hat{n} dS + B$$

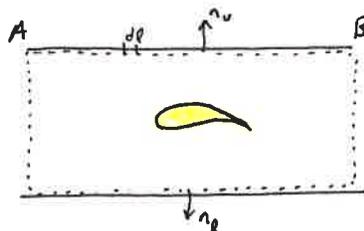
steady (with arrow pointing to the integrand)

In the z-direction

$$-Lift = \iint_S \rho v_z \hat{n}_z v_z dS + \iint_S p \hat{n}_z dS$$

Control Volume

Fixed walls (upper + lower)



Lift

$$\begin{aligned} Lift &= - \iint P_u \vec{v}_{z_u} \hat{n}_{z_u} v_{z_u} dl - \iint P_r \vec{v}_{z_r} \hat{n}_{z_r} v_{z_r} dl = - \iint P_u \vec{v}_{z_u} dl - \iint P_r \vec{v}_{z_r} dl \\ &= - \iint P_u dl + \iint P_r dl = \int_A^B (P_r - P_u) dl \end{aligned}$$

C_L Add arbitrary $\mp P_{\infty}$ integral over S

$$Lift = \int_A^B [(P_r - P_{\infty}) - (P_u - P_{\infty})] dl$$

$$C_L = \frac{Lift}{\rho c} = \frac{1}{c} \int_A^B \left(\frac{P_r - P_{\infty}}{\rho} - \frac{P_u - P_{\infty}}{\rho} \right) dl$$

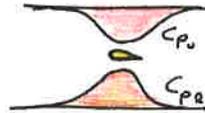
$$C_L = \frac{1}{c} \int_A^B (C_{P_r} - C_{P_u}) dl$$

What is wrong here? Why is the lift, but not the open-air lift? We made a huge assumption

Ex: The pressures on the upper and lower walls of a tunnel are:

$$C_{pe} = e^{-x^2}$$

$$C_{pu} = -e^{-x^2}$$



The airfoil has a chord of 1

What is the lift coefficient?

$$C_L = \frac{1}{c} \int_{-\infty}^{\infty} (C_{pe} - C_{pu}) dx$$

$$= \frac{1}{c} \int_{-\infty}^{\infty} e^{-x^2} + e^{-x^2} dx$$

$$= \frac{2}{c} \int_{-\infty}^{\infty} e^{-x^2} dx = \frac{2}{c} \int_{-\infty}^0 e^{-x^2} dx + \frac{2}{c} \int_0^{\infty} e^{-x^2} dx$$

$\int_0^{\infty} e^{-x^2} dx$ even function $\frac{1}{2}$

$$= \frac{4}{c} \int_0^{\infty} e^{-x^2} dx$$

Definition of "error function": $\text{erf} = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} dx$ and $\text{erf}(\infty) = 1$

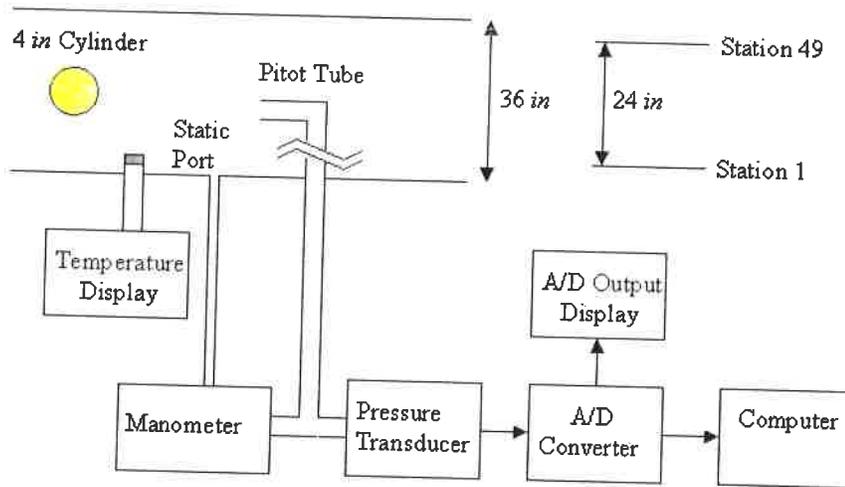
$$C_L = \frac{4}{c} \frac{\sqrt{\pi}}{2} \text{erf}(\infty)$$

$$= \frac{2\sqrt{\pi}}{c} \quad c=1$$

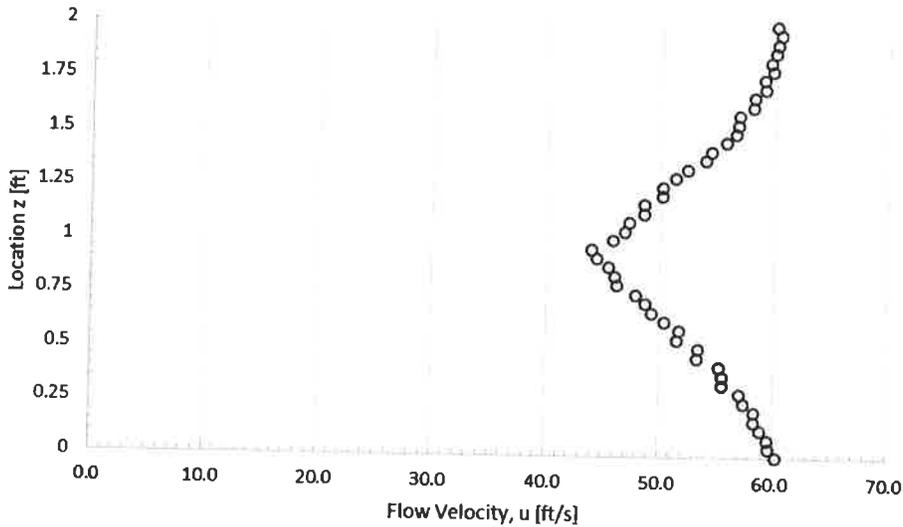
$$\boxed{C_L = 2\sqrt{\pi}}$$

Ex: Drag coefficient from numerical data

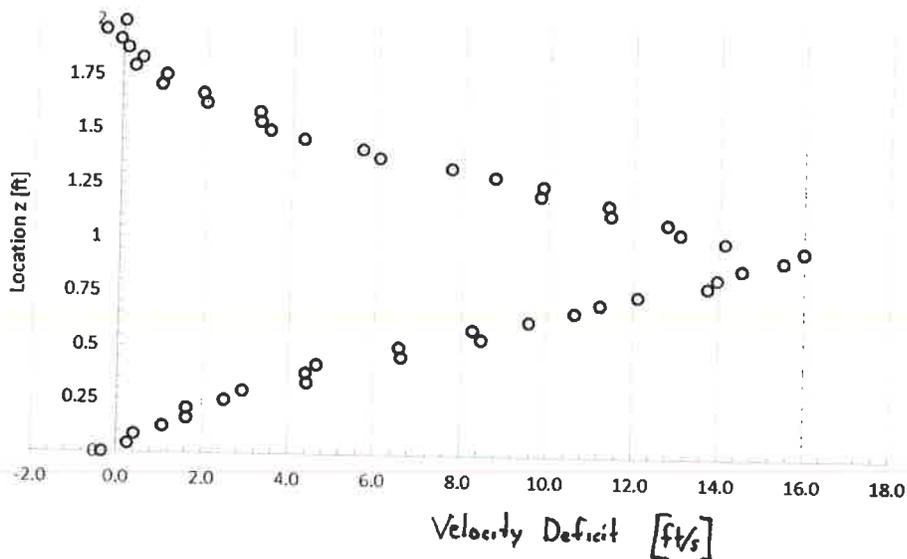
On 19th Jan 2001, your professor experimentally found the wake aft of a 4 inch diameter cylinder.



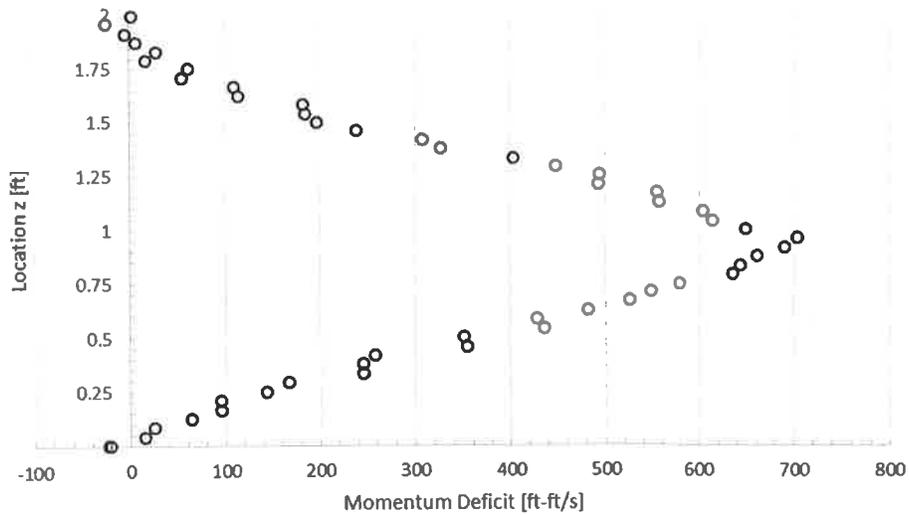
4 inch Cylinder Downstream Wake Profile



4 inch Cylinder Downstream Wake Profile



4 inch Cylinder Downstream Wake Profile



The freestream velocity = 60 ft/s

$$Re \approx 6350 \cdot 60 \frac{\text{ft}}{\text{s}} \frac{4 \text{ in}}{12 \text{ in}} = 126000$$

$$D = \int \rho_2 u_2 (u_1 - u_2) dl \Rightarrow C_d = \frac{D}{\frac{1}{2} \rho V^2 D_n} = \left(\frac{D}{P}\right) \left(\frac{2}{V^2 D_n}\right)$$

$$C_d = 1.04 \quad \text{Computed in Excel doc}$$

Slightly lower than expected. Noisy tunnel?

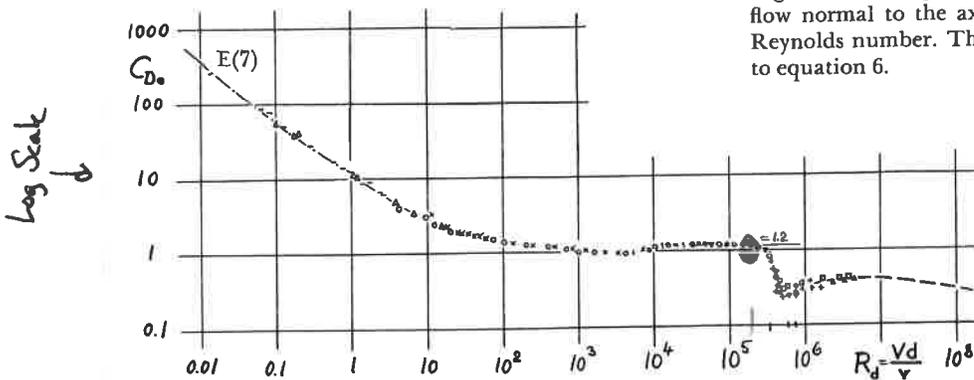
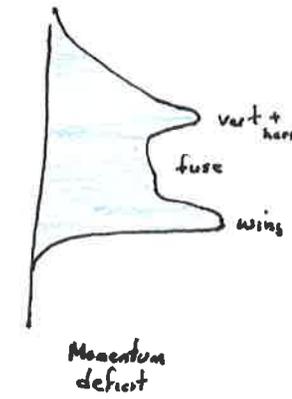
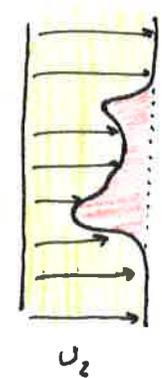
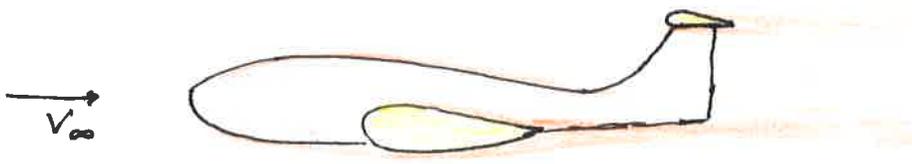


Figure 12. Drag coefficient of the circular cylinder in a flow normal to the axis (between walls), as a function of Reynolds number. The function below $R = 1$, corresponds to equation 6.

- FINN, AT LOW REYNOLDS NUMBERS (17, a)
- ▲ WHITE, WIRES FALLING IN LIQUID (17, b)
- x HELF - ARC, WIRES IN TUNNEL (17, c)
- WIESELSBERGER in WIND TUNNEL (18, a)
- ! SCHILLER-LINCKE, DROP TESTS (18, b)
- EISNER, CORRECTED FOR TURBULENCE (14)
- + N A C A, CORRECTED FOR TURBUL. (18, d)
- v GALCIT, CORRECTED FOR TURBULENCE (12)
- FECHSTEIN, IN OPEN AIR (WIND) (18, b)
- ▲ DRYDEN (NACA), IN WIND TUNNEL (8, 2)

Source: Fluid Dynamic Drag, Hoerner

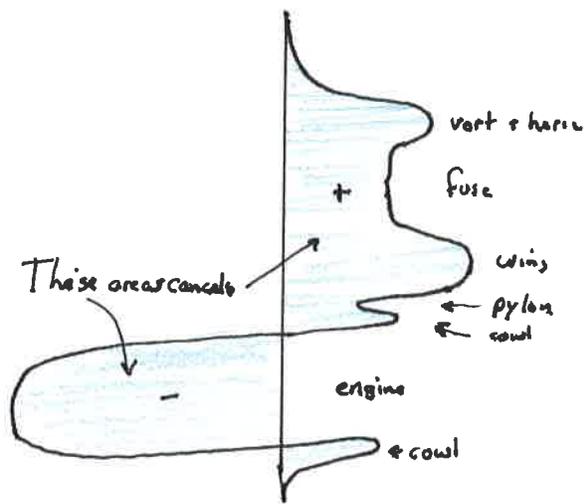
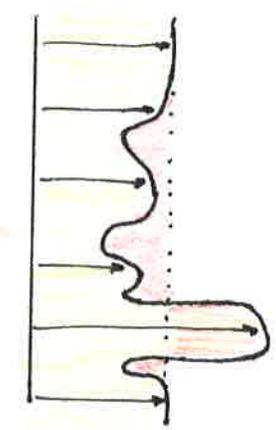
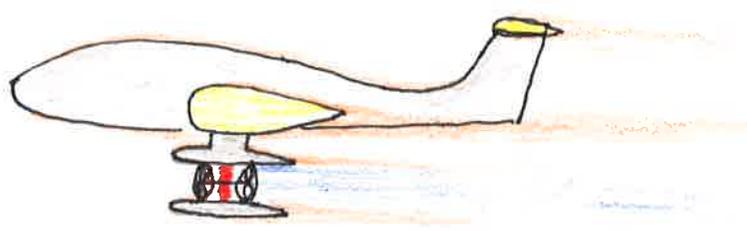
Full aircraft in steady ^{level} flight



$$C_D \geq 0$$

$F = ma \Rightarrow$ the aircraft decelerates

Propulsion!



Momentum Deficit

$$C_D = \int \frac{\rho_s U_2 (U_1 - U_2) dl}{\rho_s} = 0$$