

Lesson 7

Vorticity

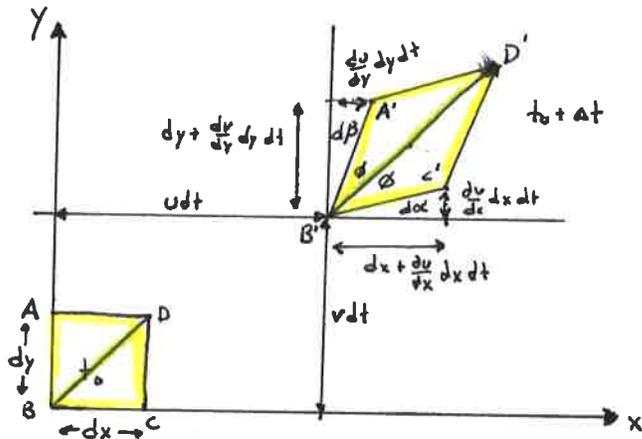
# Vorticity

Definition  $\omega = \nabla \times V$   
 $= \text{curl } V$

in 3D cartesian coordinates,

$$\omega = \left( \frac{dw}{dy} - \frac{dv}{dz} \right) \hat{i} - \left( \frac{dw}{dx} - \frac{du}{dz} \right) \hat{j} + \left( \frac{dv}{dx} - \frac{du}{dy} \right) \hat{k}$$

Distortion of a fluid element ABCD to A'B'C'D'



Define rotation as the angle of line  
 $BD \rightarrow B'D'$

$$d\Omega = \underbrace{\phi + d\alpha}_{\text{final}} - \underbrace{\frac{\pi}{4}}_{\text{initial}}$$

and

$$2\phi + d\beta + d\alpha = \frac{\pi}{2} \Rightarrow \phi = \frac{\pi}{4} - \frac{d\beta}{2} - \frac{d\alpha}{2}$$

$$d\Omega = \frac{\pi}{4} - \frac{d\beta}{2} - \frac{d\alpha}{2} + d\alpha - \frac{\pi}{4}$$

$$= \frac{d\alpha}{2} - \frac{d\beta}{2}$$

Angles

$$d\beta = \arctan\left( \frac{\frac{du}{dy} dy dt}{dy + \frac{dv}{dy} dy dt} \right) \text{ as } dt \approx 0 \text{ (formally } \lim_{dt \rightarrow 0} \text{)}$$

$$\approx \frac{\frac{du}{dy} dy dt}{dy + \frac{dv}{dy} dy dt} \cdot \frac{1}{dy dt} = \frac{\frac{du}{dy}}{\frac{1}{dt} + \frac{dv}{dy}}$$

$$\text{when } dt \rightarrow 0, \frac{1}{dt} \gg \frac{dv}{dy} \Rightarrow \frac{du}{dy} dt$$

$$d\alpha = \text{same strategy. } \approx \frac{dv}{dx} dt$$

Rotation

$$d\Omega = \frac{d\alpha}{2} - \frac{d\beta}{2} = \frac{1}{2} \frac{dv}{dx} dt - \frac{1}{2} \frac{du}{dy} dt$$

Vorticity

$$\omega \equiv 2 \frac{d\Omega}{dt} = \frac{dv}{dx} - \frac{du}{dy} = \underbrace{\nabla \times V}_{2D}$$

Vorticity is defined as twice the angular velocity

$$\omega = \nabla \times V$$

# Vorticity Transport $\omega = \nabla \times V$

- Start with momentum equation

$$\frac{\partial V}{\partial t} + V \cdot \nabla V = f - \frac{\nabla p}{\rho} + \frac{\nabla \cdot \bar{\tau}}{\rho}$$

↑ Rewrite in terms of  $\omega \Rightarrow V \cdot \nabla V = \frac{1}{2} \nabla(V \cdot V) - V \times \omega$

- Apply curl operator to entire equation ( $\nabla \times (\dots)$ )

$$\underbrace{\nabla \times \frac{\partial V}{\partial t}}_{\frac{\partial}{\partial t}(\nabla \times V)} + \frac{1}{2} \nabla \times (\nabla(V \cdot V)) - \nabla \times (V \times \omega) = \underbrace{\nabla \times f}_{f \text{ is irrot "think gravity"}} - \nabla \times \left( \frac{\nabla p}{\rho} \right) + \nabla \times \left( \frac{\nabla \cdot \bar{\tau}}{\rho} \right)$$

$\nabla \times \nabla = 0$

- Time derivative of  $\omega$  appears.

$$\frac{d\omega}{dt} - \nabla \times (V \times \omega) = - \nabla \times \left( \frac{\nabla p}{\rho} \right) + \nabla \times \left( \frac{\nabla \cdot \bar{\tau}}{\rho} \right)$$

- $\nabla \times (V \times \omega) = \underbrace{V \nabla \cdot \omega}_{\nabla \cdot (\nabla \times V)!!} - \omega \nabla \cdot V + \omega \cdot \nabla V - V \cdot \nabla \omega$

- $\nabla \times \left( \frac{\nabla p}{\rho} \right)$  Identity says  $\nabla \times (cA) = c(\nabla \times A) + (\nabla c) \times A$

thus  $\nabla \times (\bar{\rho}^{-1} \nabla p) = \bar{\rho}^{-1} \nabla \times \nabla p + (\nabla \bar{\rho}^{-1}) \times \nabla p$

and  $\nabla \bar{\rho}^{-1} = \nabla \left( \frac{f}{g} \right) = \frac{g \nabla f - f \nabla g}{g^2} = \frac{\rho \nabla \bar{\tau} - |\nabla p|}{\rho^2}$

$$\nabla \times \left( \frac{\nabla p}{\rho} \right) = - \frac{\nabla \rho \times \nabla p}{\rho^2}$$

- $\nabla \times \left( \frac{\nabla \cdot \bar{\tau}}{\rho} \right)$  leave it alone.

$$\frac{d\omega}{dt} + \omega \nabla \cdot V - \omega \cdot \nabla V + V \cdot \nabla \omega = \frac{\nabla \rho \times \nabla p}{\rho^2} + \nabla \times \left( \frac{\nabla \cdot \bar{\tau}}{\rho} \right)$$

a

Eulerian Frame

$$\frac{d\omega}{dt} - \nabla \times (V \times \omega) = \frac{\nabla \rho \times \nabla p}{\rho^2} + \nabla \times \left( \frac{\nabla \cdot \bar{\tau}}{\rho} \right)$$

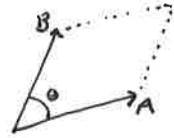
Lagrangian Frame

$$\frac{D\omega}{Dt} \left( \frac{\omega}{\rho} \right) = \frac{\omega}{\rho} \cdot \nabla V + \frac{\nabla \rho \times \nabla p}{\rho^3} + \frac{1}{\rho} \nabla \times \left( \frac{\nabla \cdot \bar{\tau}}{\rho} \right)$$

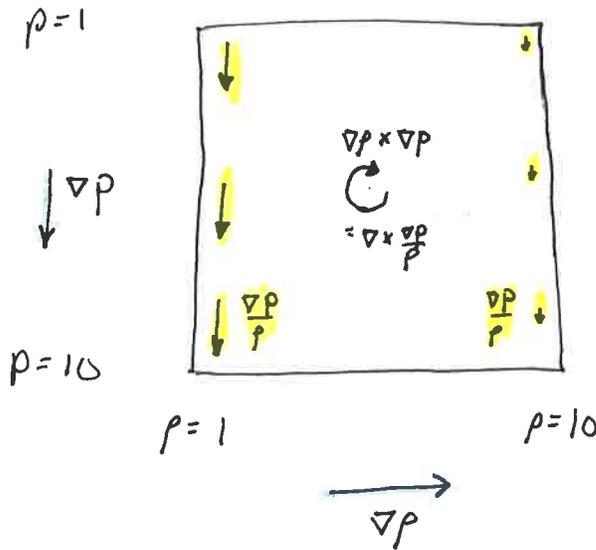
# Baroclinic

$$\nabla \rho \times \nabla p$$

Remember  $A \times B = |A||B| \sin \theta$



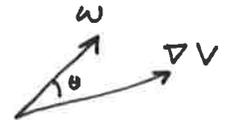
If the density and pressure gradients are not aligned, vorticity is generated.



$$\nabla \rho \times \nabla p \propto -\nabla \times \left( \frac{\nabla p}{\rho} \right)$$

# Vortex Stretching

$$\omega \cdot \nabla V = |\omega| |\nabla V| \cos \theta$$



Conservation of angular momentum.

Vorticity increases when the vorticity vector is aligned with accelerating flow

Ex: Certain B737 models have a cowling mounted vortex generator located such that the circulation travels over the wing. Where is  $|\omega|$  greatest? Assume subsonic flow.



A:  $\nabla V \leftarrow \Rightarrow \frac{d|\omega|}{dt} < 0$

B:  $\nabla V \rightarrow \Rightarrow \frac{d|\omega|}{dt} > 0$

C:  $\nabla V \approx 0 \Rightarrow \frac{d|\omega|}{dt} = 0$

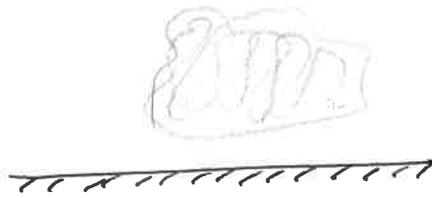
D:  $\nabla V \approx 0 \Rightarrow \frac{d|\omega|}{dt} = 0$

$|\omega|$  greatest where  $\frac{d}{dt} (|\omega| |\nabla V| \cos \theta) = 0$

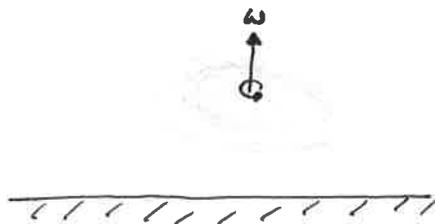
**C**

# Vortex Stretching

- Say we have an airmass



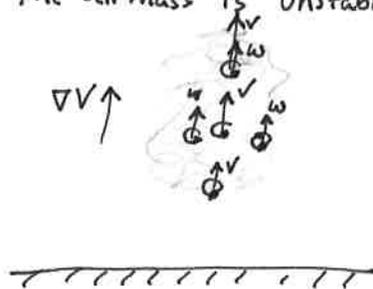
- Through some process, it ~~now~~ now has vorticity (of a low magnitude)



perhaps  $\nabla \rho \times \nabla p \neq 0$

$\nabla \cdot \vec{v}$  and  $\nabla p$  are not aligned.  
in the boundary layer

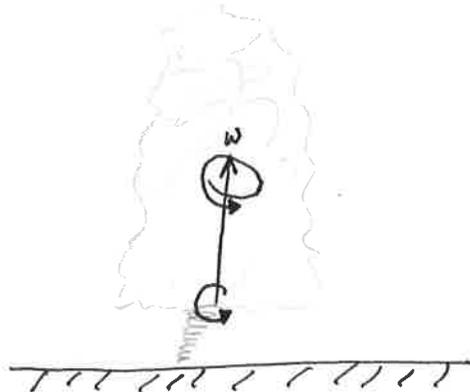
- Given that the airmass is unstable (hot humid air), the air accelerates upward



$$\begin{aligned} \frac{D\omega}{Dt} &= \omega \cdot \nabla V + v \nabla \omega \\ &= |\omega| |\nabla V| \cos \theta \end{aligned}$$

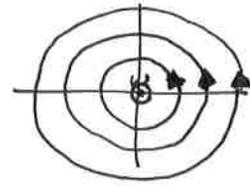
$\theta$  small

- Vortex stretching increases the vorticity



Given a region of vorticity where circular streamlines exist ~~and vorticity is constant~~ such that  $\omega \cdot \nabla V = 0$ , the governing equation is

$$\frac{D\omega}{Dt} = \cancel{\omega \cdot \nabla V} + \nu \nabla^2 \omega$$



$\omega$  is a vector out of the page.

How does the vorticity behave?

$$\frac{D\omega}{Dt} = \nu \left( \frac{d^2\omega}{dx^2} + \frac{d^2\omega}{dy^2} \right) = \nu \left( \frac{d^2\omega}{dr^2} + \frac{1}{r} \frac{d\omega}{dr} + \frac{1}{r^2} \frac{d^2\omega}{d\theta^2} \right)$$

This is a PDE with a classic Separation of Variables approach.

$$\omega = T R \Theta$$

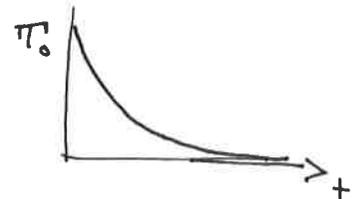
Subst.

$$\frac{T'}{T} = \nu \left( \frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{\Theta''}{\Theta} \right) = -\lambda$$

Thus, in time

$$\frac{1}{\nu} \frac{dT}{dt} = -\lambda T \Rightarrow \frac{dT}{dt} + \nu \lambda T = 0$$

$$\text{Solution is } T = T_0 e^{-\nu \lambda t}$$



Vorticity decays to zero exponentially (eventually)