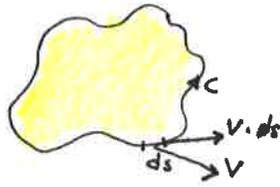


Lesson 8
Circulation

Circulation Γ

Definition

$$\Gamma = - \oint_C \mathbf{V} \cdot d\mathbf{s}$$

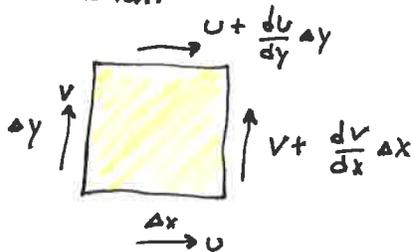


English

Circulation is proportional to the velocity component tangent to a closed curve.

Related to vorticity...

• Take a small element



• Definition of Γ with a path C as the edges of the element (ccw)

$$\Delta \Gamma = - \oint_C \mathbf{V} \cdot d\mathbf{s} = - \int_1 \mathbf{V} \cdot d\mathbf{s} + - \int_2 \mathbf{V} \cdot d\mathbf{s} + \int_3 \dots + \int_4 \dots$$



$$= - \int U \Delta x - \int (v + \frac{dv}{dx} \Delta x) \Delta y - \int (U + \frac{du}{dy} \Delta y) (\Delta x) (-1) - \int v \Delta y (-1)$$

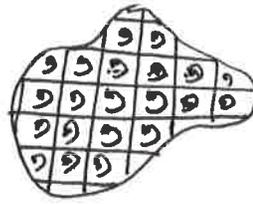
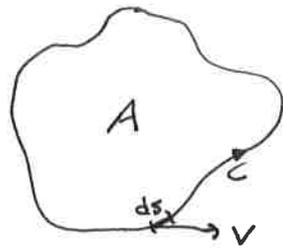
all terms are constant

$$= -U \Delta x - v \Delta y - \frac{dv}{dx} \Delta x \Delta y + U \Delta x + \frac{du}{dy} \Delta y \Delta x + v \Delta y$$

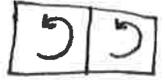
$$\Delta \Gamma = \left(\frac{du}{dy} - \frac{dv}{dx} \right) \Delta x \Delta y$$

This is $\nabla \times \mathbf{V}$ in 2D !!

- Expand to region using summation of areas



since



↑
Integral along neighbor cancel.

$$\oint V \cdot ds = \sum \Delta \Gamma = \sum \omega \Delta A$$

$$\Gamma = - \oint_C V \cdot ds = - \iint_A \omega \cdot \hat{n} dA$$

Circulation is the integral of vorticity.

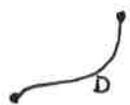
Irrotational Flow

When $\omega \equiv \nabla \times V = 0$, the flow is irrotational.

Irrotational flows have zero circulation since $\Gamma = - \iint_{\hat{A}} \omega \cdot \hat{n} dA = 0$

Also, the term $\int_C V \cdot ds$ is independent of path, since $\Gamma = - \oint V \cdot ds$

Ex: $\int_C V \cdot ds = 0$ when $\nabla \times V = 0$. Is the open curve D integral zero? ^{always; even}

$\int_D V \cdot ds \stackrel{?}{=} 0$ where 

No, only a closed curve in an irrotational flow is always zero.

From math, if $\int_C V \cdot ds = 0$ regardless of path, then $V \cdot ds = d\phi$ where ϕ is a scalar field of unique value (i.e. function).

Expand $V \cdot ds = d\phi$ where $ds = (dx, dy, dz)$ and $d\phi = \frac{d\phi}{dx} dx + \frac{d\phi}{dy} dy + \frac{d\phi}{dz} dz$

$u dx + v dy + w dz = \frac{d\phi}{dx} dx + \frac{d\phi}{dy} dy + \frac{d\phi}{dz} dz$



 these terms match!

$u = \frac{d\phi}{dx} \quad v = \frac{d\phi}{dy} \quad w = \frac{d\phi}{dz}$

In general, $V = \nabla \phi$

We call $\nabla \phi$ the velocity potential. Notice that $\omega = \nabla \times V$
 $= \nabla \times (\nabla \phi) = 0$

All irrotational flows have a corresponding potential field. Thus, an irrotational flow is often called a potential flow. The curl of a scalar gradient is zero.

If vorticity exists in a flow, ~~an~~ the ϕ field is no longer single valued.

If incompressible, $\nabla \cdot V = 0$ thus $\nabla \cdot \nabla \phi = 0$ thus $\nabla^2 \phi = 0$ (more later!)

Kelvin's Theorem

Lord Kelvin: 1824 - 1907 Temperature unit named after him

He took the name Kelvin after the river Kelvin flowing through Scotland.

Pick a curve (closed) along a closed fluid path



Take time derivative of this circulation

$$-\frac{d\Gamma}{dt} = \frac{d}{dt} \left(\oint_C \mathbf{V} \cdot d\mathbf{s} \right) \Rightarrow \text{chain rule} \Rightarrow -\frac{d\Gamma}{dt} = \oint_C \frac{d\mathbf{V}}{dt} \cdot d\mathbf{s} + \oint_C \mathbf{V} \cdot \frac{d}{dt}(d\mathbf{s})$$

The momentum eqn for an inviscid flow is

$$\frac{d\mathbf{V}}{dt} = \mathbf{f} - \frac{1}{\rho} \nabla p \quad \text{when } \mathbf{f} = -\nabla F \text{ (i.e. irrotational force)} \quad \frac{d\mathbf{V}}{dt} = \nabla F - \frac{1}{\rho} \nabla p$$

Also, here.

$$\frac{d}{dt}(d\mathbf{s}) = d\left(\frac{d\mathbf{s}}{dt}\right) = d(\mathbf{V}) = d\mathbf{V}$$

Substitute:

$$-\frac{d\Gamma}{dt} = \oint_C -\nabla F \cdot d\mathbf{s} - \oint_C \frac{1}{\rho} \nabla p \cdot d\mathbf{s} + \oint_C \mathbf{V} \cdot d\mathbf{V}$$

$$-\frac{d\Gamma}{dt} = \underbrace{\oint_C \mathbf{f} \cdot d\mathbf{s}}_{\substack{0 \text{ when} \\ \mathbf{f} \text{ is} \\ \text{conservative} \\ \text{(gravity)}}} - \underbrace{\oint_C \frac{1}{\rho} \nabla p \cdot d\mathbf{s}}_{\substack{\text{Exact differential} \\ \text{when } p = f(p)}} + \underbrace{\oint_C \mathbf{V} \cdot d\mathbf{V}}_{\substack{\text{Exact differential} \\ \mathbf{V} \cdot d\mathbf{V} = u du + v dv + w dw \\ \text{over a closed curve} = 0}}$$

$$= 0$$

In a flow field without viscosity, a conservative body force, and a barotropic density, the circulation does not change in time.

$$\boxed{\frac{d\Gamma}{dt} = 0}$$

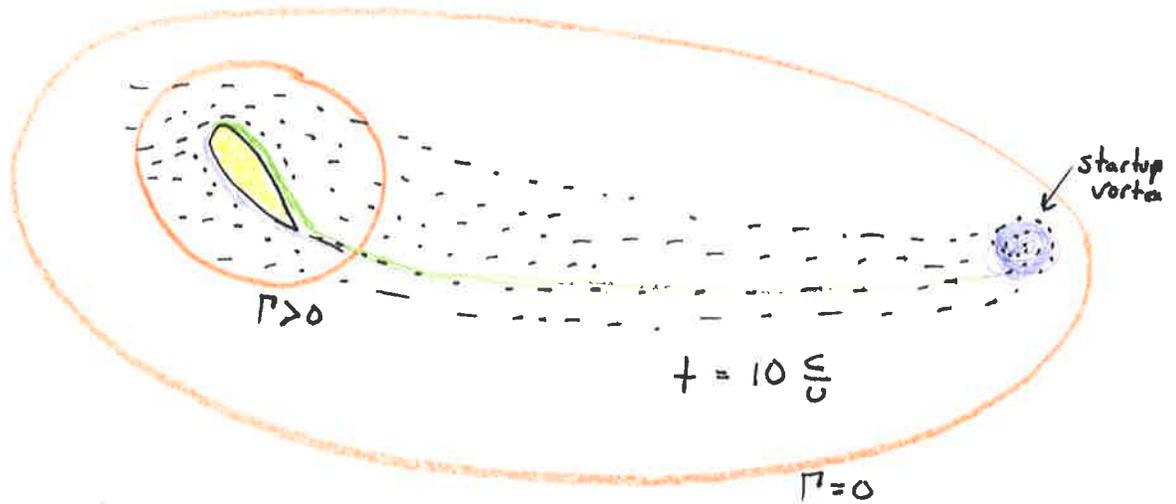
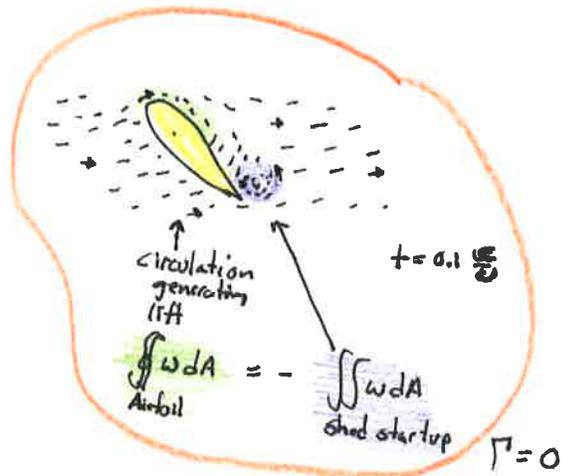
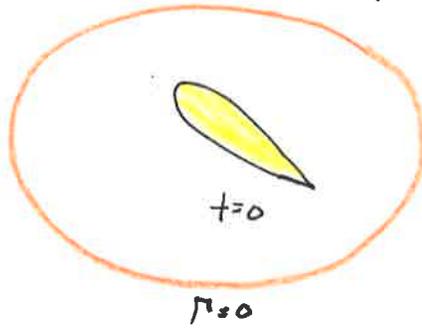
Nothing was restricted regarding ω (other than inviscid flow...), so this is valid for the pressure field (i.e. Euler terms) of the governing equations.

Kelvins Theorem. (continued)

If $\frac{d\Gamma}{dt} = 0$, then Γ in a flow is constant.

Thus, any inviscid vorticity generated must have an equal and opposite circulation elsewhere.

Ex. Impulsive start of an airfoil



Duality of Velocity and Vorticity - Dilatation (aka. Vorticity - Source)

By now, you've seen that vorticity and dilatation are just operations on velocity. ω and σ are just special ways to look at velocity.

$$\omega = \nabla \times V \quad \sigma = \nabla \cdot V$$

What about the inverse,

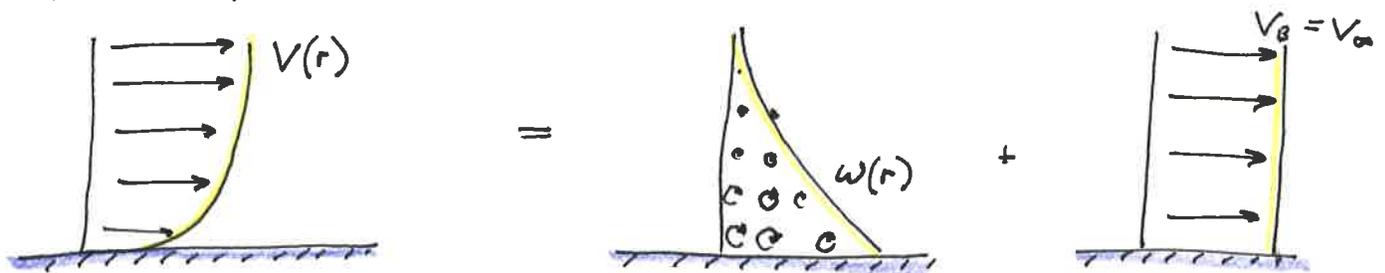
$$V = V_\sigma + V_\omega + V_b$$

$$V_\sigma \equiv \frac{1}{4\pi} \iiint \sigma(r') \frac{r-r'}{|r-r'|^3} dx' dy' dz'$$

$$V_\omega \equiv \frac{1}{4\pi} \iiint \omega(r') \times \frac{r-r'}{|r-r'|^3} dx' dy' dz'$$

$$V_b = V_\infty$$

Ex: Boundary Layer



No approximation. This is exactly dual.

Dilatation Transport $\sigma = \nabla \cdot V$

- Start with momentum equ.

$$\underbrace{\frac{dV}{dt} + V \cdot \nabla V}_{\frac{DV}{DT}} = f - \frac{\nabla P}{\rho} + \frac{\nabla \cdot \bar{\tau}}{\rho}$$

- Apply div

$$\underbrace{\nabla \cdot \frac{DV}{DT}}_{\frac{D}{Dt}(\nabla \cdot V)} = \nabla \cdot f - \nabla \cdot \left(\frac{\nabla P}{\rho} \right) + \nabla \cdot \left(\frac{\nabla \cdot \bar{\tau}}{\rho} \right)$$

For a constant density flow (incompressible)

$$\frac{D\sigma}{Dt} = \nabla \cdot f - \frac{1}{\rho} \nabla^2 P + \bar{\tau} \left(\frac{d^2 \tau_{xx}}{dx^2} + 2 \frac{d^2 \tau_{xy}}{dx dy} + 2 \frac{d^2 \tau_{xz}}{dx dz} + \frac{d^2 \tau_{yy}}{dy^2} + \frac{d^2 \tau_{zz}}{dz^2} + 2 \frac{d^2 \tau_{yz}}{dy dz} \right)$$

web.mit.edu/drela/Public/web/xfoil



XFOIL is an interactive program for the design and analysis of subsonic isolated airfoils.

It consists of a collection of menu-driven routines which perform various useful functions such as:

- Viscous (or inviscid) analysis of an existing airfoil, allowing
 - forced or free transition
 - transitional separation bubbles
 - limited trailing edge separation
 - lift and drag predictions just beyond CL_{max}
 - Karman-Tsien compressibility correction
 - fixed or varying Reynolds and/or Mach numbers
- Airfoil design and redesign by interactive modification of surface speed distributions, in two methods:
 - Full-Inverse method, based on a complex-mapping formulation
 - Mixed-Inverse method, an extension of XFOIL's basic panel method
- Airfoil redesign by interactive modification of geometric parameters such as
 - max thickness and camber, highpoint position
 - LE radius, TE thickness
 - camber line via geometry specification
 - camber line via loading change specification
 - flap deflection
 - explicit contour geometry (via screen cursor)
- Blending of airfoils
- Writing and reading of airfoil coordinates and polar save files
- Plotting of geometry, pressure distributions, and multiple polars

Release Conditions