

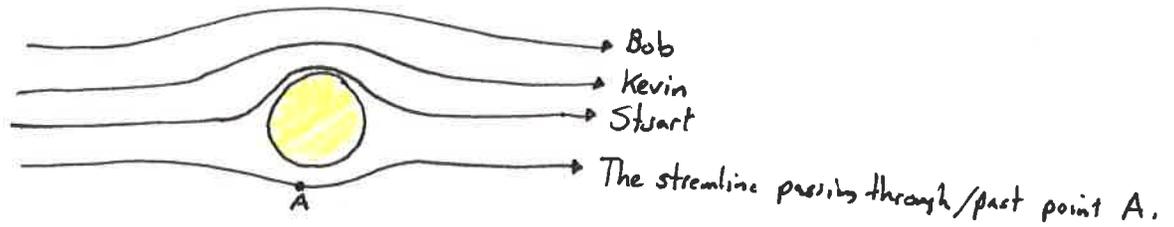
Lesson 10

Streamfunction

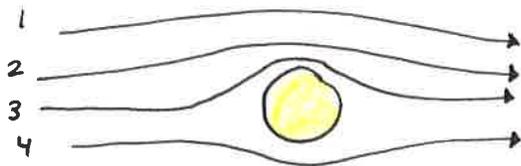
ψ

Given a flow field with streamlines, how can we organize them?

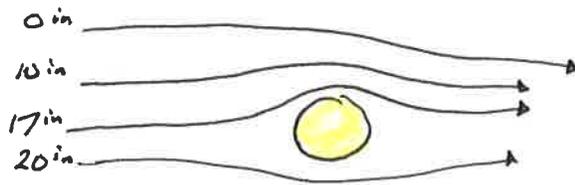
• Name:



• Number



• Distance



• Mass flow (Canonical method)

$$\dot{m} = \rho V dn$$

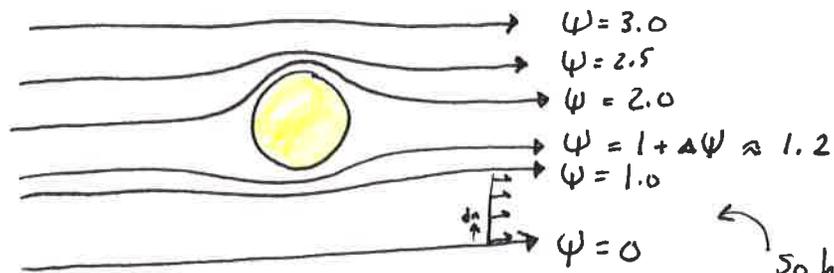
$$\frac{V}{\rho} \rightarrow \frac{d\psi}{dn}$$

Units

$$\rho V dn = \frac{\text{slug}}{\text{ft}^3} \frac{\text{ft}}{\text{s}} \text{ft}$$

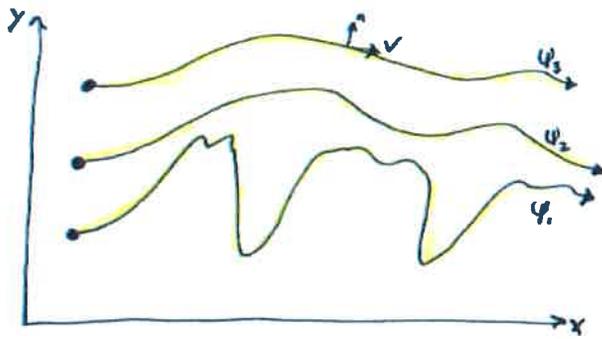
$$= \frac{\text{slug}}{\text{s}} \frac{1}{\text{ft}}$$

mass flow rate unit depth



So between $\psi = 0$ and $\psi = 1$, $\frac{\dot{m}}{\text{depth}} = 1$

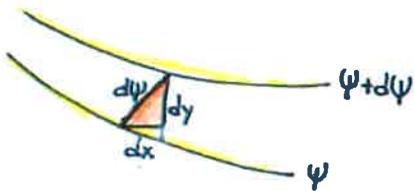
Stream function



Streamlines are defined as lines which no fluid crosses. ($\mathbf{v} \cdot \mathbf{n} = 0$)

The difference in magnitude between two stream functions provides a measure of the volume of fluid flowing between the two streamlines per second (per unit depth).
 ψ [ft³/s]

Pick a control volume between two streamlines (ψ and $\psi + d\psi$)



Flow into CV = $d\psi$

Flow out of CV = $u dy - v dx$

↑ negative since normal for dx side is $-\hat{j}$

Definition of an exact differential of ψ

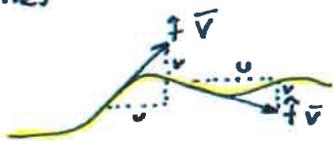
$$d\psi = \frac{d\psi}{dx} dx + \frac{d\psi}{dy} dy$$

Match terms

$$\frac{d\psi}{dx} dx + \frac{d\psi}{dy} dy = d\psi = u dy - v dx$$

$$\Rightarrow u = \frac{d\psi}{dy} \quad \text{and} \quad v = -\frac{d\psi}{dx}$$

Streamlines



"The streamline is a curve whose tangent at every point coincides with the direction of the velocity vector"

Bortin-Smith

$$\frac{dx}{dy} = \frac{u}{v} \Rightarrow \underbrace{u dy - v dx}_{= d\psi} = 0$$

Div and Curl (incompressible, irrotational)

$$\nabla \cdot \mathbf{v} \stackrel{?}{=} 0 \Rightarrow \frac{dv}{dx} + \frac{du}{dy} = \frac{d^2\psi}{dx dy} - \frac{d^2\psi}{dy dx} = 0 \quad \checkmark$$

we saw this term above = $d\psi$

Thus along a streamline, no flow crosses the curve.

$$\nabla \times \mathbf{v} \stackrel{?}{=} 0 \Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u & v & w \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \frac{dv}{dx} - \frac{du}{dy} = \frac{d}{dx} \left(-\frac{d\psi}{dx} \right) - \frac{d}{dy} \left(\frac{d\psi}{dy} \right) = \nabla^2 \psi = 0$$

ψ is also a Laplacian for incomp + irrot.

PDE Theory

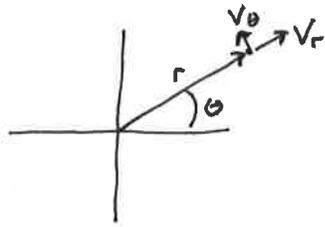
$\nabla^2 \psi = 0$ is ~~possible~~ a linear PDE. Any combination of solutions can be added together!!

In fact, $F = \phi + i\psi$

$$\nabla^2 F = 0$$

$$\psi_{\text{total}} = \psi_1 + \psi_2 + \psi_3 + \dots$$

Polar Coordinates:



$$\rho V_r = \frac{1}{r} \frac{d\psi}{d\theta}$$
$$\rho V_\theta = -\frac{d\psi}{dr}$$

Conversion to V_x and V_y :

$$\begin{aligned} V_x &= V_r \cos \theta - V_\theta \sin \theta \\ V_y &= V_r \sin \theta + V_\theta \cos \theta \end{aligned} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{pmatrix} V_r \\ V_\theta \end{pmatrix}$$

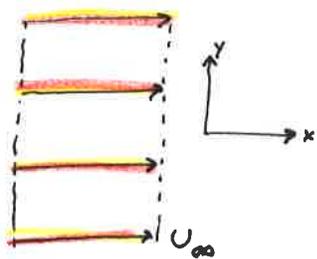
Incompressible

Density is constant.

$$\begin{aligned} u &= \frac{d\psi}{dy} & \text{and} & & V_r &= \frac{1}{r} \frac{d\psi}{d\theta} \\ v &= -\frac{d\psi}{dx} & & & V_\theta &= -\frac{d\psi}{dr} \end{aligned}$$

Remember that the streamfunction is for 2D steady flows.

Uniform Flow



Flow moving in a fixed direction at a uniform speed

Derivation:

$$u = U_{\infty} \quad \text{and} \quad \frac{d\psi}{dy} = u$$

$$v = 0 \quad \text{and} \quad \frac{d\psi}{dx} = -v$$

Substitute $u = U_{\infty}$ into $\frac{d\psi}{dy} = u$

$$\frac{d\psi}{dy} = U_{\infty} \quad \text{and} \quad \frac{d\psi}{dx} = 0$$

Integrate

$$d\psi = U_{\infty} dy \quad \text{y direction}$$

$$d\psi = 0 dx \quad \text{x direction}$$

Solution y

$$\int d\psi = \int U_{\infty} dy = U_{\infty} y \Big|_0^y$$

$$\psi(y) - \underbrace{\psi(0)}_{\text{arbitrary starting pt}} = U_{\infty} y$$

$$\psi = U_{\infty} y$$

in y terms

Solution x

$$\int d\psi = 0 \Rightarrow \psi = 0 \quad \text{in x terms}$$

$$\psi = U_{\infty} y$$

$$= U_{\infty} r \sin \theta$$

← Lines of constant y

Velocity potential

$$V = \nabla \phi \Rightarrow u = \frac{d\phi}{dx} \quad v = \frac{d\phi}{dy}$$

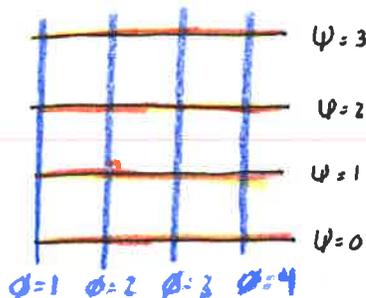
$$\text{Thus, } u = U_{\infty} = \frac{d\phi}{dx} \Rightarrow \phi = U_{\infty} x$$

$$v = 0 = \frac{d\phi}{dy} \Rightarrow \phi = 0$$

$$\phi = U_{\infty} x$$

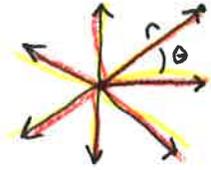
$$= U_{\infty} r \cos \theta$$

← lines of constant x



Source

Flow emanates from a point



Streamlines are at a constant angle.

$$\psi = C \cdot \theta \quad \text{where} \quad \theta = \text{atan}\left(\frac{y}{x}\right)$$

The canonical form is

$$\text{ATAN2}(x, y)$$

$$\begin{aligned} \psi &= \frac{\Lambda}{2\pi} \theta \\ &= \frac{\Lambda}{2\pi} \text{atan}\left(\frac{y}{x}\right) \end{aligned}$$

Ex: Given the ψ for a source, find the velocity potential

Work in polar coordinates since $\psi(\theta)$

$$V_r = \text{radial velocity} = \frac{1}{r} \frac{d\psi}{d\theta} = \frac{1}{r} \frac{d}{d\theta} \left(\frac{\Lambda}{2\pi} \theta \right) = \frac{\Lambda}{2\pi r}$$

$$V_\theta = \text{"theta" velocity} = -\frac{d\psi}{dr} = -\frac{d}{dr} \left(\frac{\Lambda}{2\pi} \theta \right) = 0$$

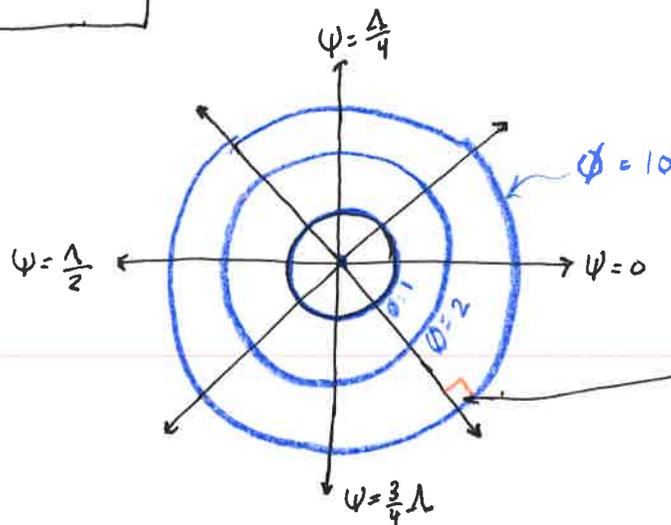
Definition of velocity potential ϕ in polar coordinates

$$V_r = \frac{d\phi}{dr} \quad \text{and} \quad V_\theta = \frac{1}{r} \frac{d\phi}{d\theta}$$

$$V_r = \frac{d\phi}{dr} = \frac{1}{r} \frac{d\psi}{d\theta} = \frac{\Lambda}{2\pi r} \quad \Rightarrow \int d\phi = \int \frac{\Lambda}{2\pi} \frac{dr}{r} \quad \Rightarrow \phi = \frac{\Lambda}{2\pi} \ln r$$

$$V_\theta = \frac{1}{r} \frac{d\phi}{d\theta} = 0 \quad \Rightarrow d\phi = 0$$

$$\boxed{\phi = \frac{\Lambda}{2\pi} \ln r} \quad \leftarrow \text{lines of constant radius.}$$



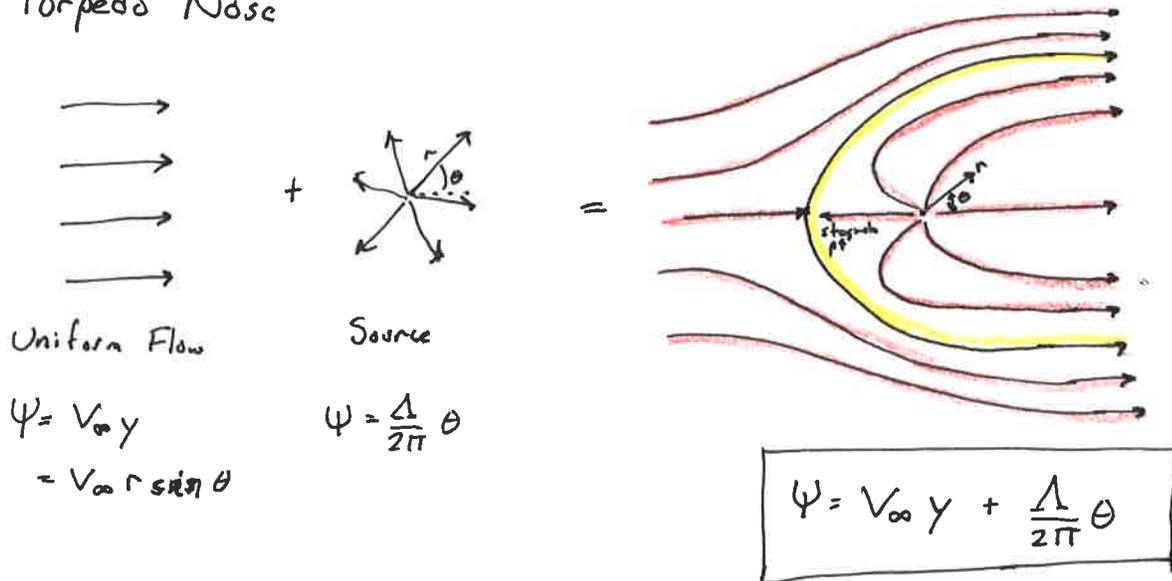
Do you notice anything interesting about the intersection of ψ and ϕ ?

Linear Elementary Flows.

Since $\nabla^2 \psi = 0$ and PDE theory says $\psi = \psi_1 + \psi_2 + \dots$

We can directly add our stream functions to make new flows

Ex: Torpedo Nose



Velocity?

$$V_r = \frac{1}{r} \frac{d\psi}{d\theta} = \frac{1}{r} \frac{d}{d\theta} \left(V_\infty r \sin \theta + \frac{\Lambda}{2\pi} \theta \right) = \frac{1}{r} \left(V_\infty r \cos \theta + \frac{\Lambda}{2\pi} \right)$$

$$V_\theta = -\frac{d\psi}{dr} = -V_\infty \sin \theta = V_\infty \cos \theta + \frac{\Lambda}{2\pi r}$$

Notice that the velocities are also the addition since a derivative is a linear operation!!

Where is the stagnation pt?

$$V_\theta = V_r = 0 \quad \Rightarrow \quad V_r = V_\infty \cos \theta + \frac{\Lambda}{2\pi r} = 0$$

and

$$V_\theta = -V_\infty \sin \theta = 0 \quad \leftarrow \text{either } V_\infty = 0 \text{ or } \sin \theta = 0 \Rightarrow \theta = 0 \text{ or } \pi$$

Thus $V_r = 0 = V_\infty \cos \theta + \frac{\Lambda}{2\pi r} \Rightarrow r = \frac{-\Lambda}{2\pi V_\infty \cos \theta}$

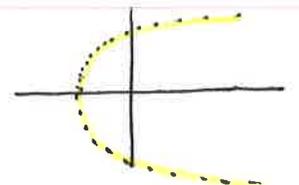
What is the shape of the nose?

Pick the streamline passing through the stagnation point above.

$$\psi_{sp} = V_\infty y + \frac{\Lambda}{2\pi} \theta \quad \text{but } y = 0 \text{ and } \theta = \pi$$

$$= \frac{\Lambda}{2\pi} \pi = \frac{\Lambda}{2} \Rightarrow \psi = \frac{\Lambda}{2} = V_\infty r \sin \theta + \frac{\Lambda}{2\pi} \theta$$

pick all pts where $\psi = \frac{\Lambda}{2}$



What is the pressure coefficient?

$$C_p = \frac{P - P_\infty}{\rho} \quad \text{Assuming } B', \quad \rho + \frac{1}{2}\rho V^2 = \text{Constant} \quad \text{and} \quad P(V_\infty) = P_\infty = P_\infty + \frac{1}{2}\rho V_\infty^2$$

$$= \frac{P_\infty + \frac{1}{2}\rho V_\infty^2 - \frac{1}{2}\rho V^2 - P_\infty}{\frac{1}{2}\rho V_\infty^2} = \frac{\frac{1}{2}\rho V_\infty^2 - \frac{1}{2}\rho V^2}{\frac{1}{2}\rho V_\infty^2} = 1 - \frac{V^2}{V_\infty^2}$$

Incompressible ONLY

$$V^2 = V_r^2 + V_\theta^2 = \left(V_\infty \cos \theta + \frac{\Delta}{2\pi r} \right)^2 + \left(-V_\infty \sin \theta \right)^2$$

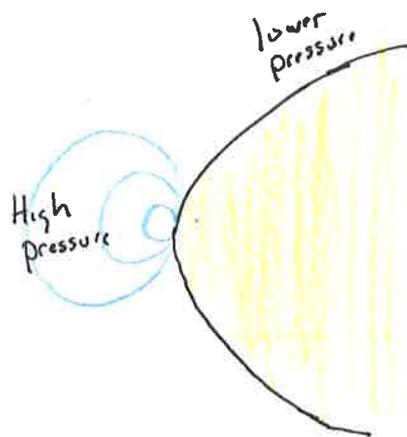
$$= V_\infty^2 \cos^2 \theta + \frac{\Delta}{\pi r} V_\infty \cos \theta + \frac{\Delta^2}{(2\pi r)^2} + V_\infty^2 \sin^2 \theta$$

$V_\infty^2 \cdot 1$

$$V^2 = V_\infty^2 + \frac{\Delta}{\pi r} V_\infty \cos \theta + \frac{\Delta^2}{4\pi^2 r^2}$$

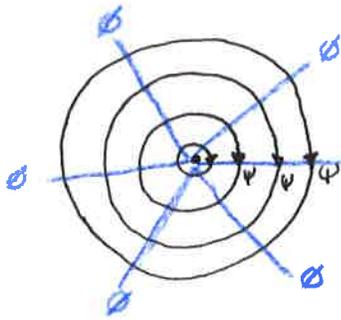
$$C_p = 1 - \frac{V^2}{V_\infty^2} = 1 - \frac{V_\infty^2 + \frac{\Delta}{\pi r} V_\infty \cos \theta + \frac{\Delta^2}{4\pi^2 r^2}}{V_\infty^2} = -\frac{\Delta}{V_\infty} \frac{1}{\pi r} \cos \theta - \frac{\Delta^2}{4\pi^2 r^2 V_\infty^2}$$

and $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$



Vortex

Irrrotational flow about a point.



$$\psi = \frac{\Gamma}{2\pi} \ln r$$

$$\phi = -\frac{\Gamma}{2\pi} \theta$$

Where have you seen these before?

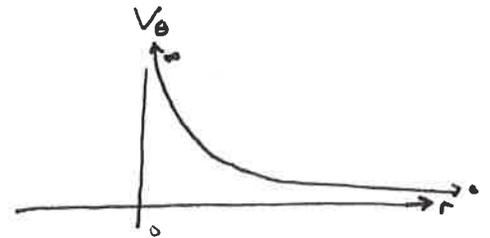
ψ_{vortex} has the same function as ϕ_{source}

ϕ_{vortex} " " " ψ_{source}

What is the velocity profile?

$$V_r = \frac{1}{r} \frac{d\psi}{d\theta} = \frac{1}{r} \frac{d}{d\theta} \left(\frac{\Gamma}{2\pi} \ln r \right) = 0$$

$$V_\theta = -\frac{d\psi}{dr} = -\frac{d}{dr} \left(\frac{\Gamma}{2\pi} \ln r \right) = -\frac{\Gamma}{2\pi} \frac{1}{r}$$



Since $V_\theta(r=0) = \infty$, the vortex is only a theoretical construct. But it is useful as a model of flow.

Are you sure that the vortex is irrotational? It is rotating!!

Irrrotational means zero vorticity. $\omega = \nabla \times V = 0$

$$\text{Curl in polar coordinates is } \omega_z = \frac{1}{r} \frac{d}{dr} (r V_\theta) - \frac{1}{r} \frac{d}{d\theta} (V_r)$$

$$= \frac{1}{r} \frac{d}{dr} \left(r \cdot \left(-\frac{\Gamma}{2\pi} \right) \frac{1}{r} \right) - \frac{1}{r} \frac{d}{d\theta} (0)$$

$$= \frac{1}{r} \frac{d}{dr} \left(-\frac{\Gamma}{2\pi} \right) = 0 \quad \checkmark$$

What is happening? the $r V_\theta$ term for rotation is exactly cancelling with the velocity profile $\frac{1}{r}$.

What happens at $r=0$? $\frac{1}{r}$ is undefined. We remove this point!