

Lesson 11 part 1

Joukowsky

# Joukowski Transform (circle to airfoil)

We want but potential flow theory gives us circles

Remember that both  $\phi$  and  $\psi$  were shown to satisfy  $\nabla^2 F = 0$  where  $F = \phi + i\psi$

Conformal Transforms with  $\nabla^2 F = 0$

Complex # theory shows that if  $\nabla^2 F = 0$ , then any analytic function mapping also has  $\nabla^2 F = 0$

Ex:

$$f(z) = f(x+iy) \Rightarrow z = x+iy$$

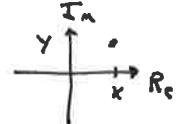
$$\frac{df}{dx} = \frac{df}{dz} \frac{dz}{dx} = \frac{df}{dz} \cdot 1 = \frac{df}{dz} \quad \text{and} \quad \frac{df}{dy} = \frac{df}{dz} \frac{dz}{dy} = i \frac{df}{dz}$$

$$\frac{d}{dx} \left( \frac{df}{dx} \right) = \frac{d}{dz} \left( \frac{df}{dx} \right) \frac{dz}{dx} = \frac{d}{dz} \left( \frac{df}{dz} \right) = \frac{d^2 f}{dz^2}$$

$$\frac{d}{dy} \left( \frac{df}{dy} \right) = \frac{d}{dz} \left( \frac{df}{dy} \right) \frac{dz}{dy} = \frac{d}{dz} \left( i \frac{df}{dz} \right) i = i^2 \frac{df}{dz} = -\frac{d^2 f}{dz^2}$$

Apply to Laplace eqn

$$\nabla^2 F = 0 = \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} = \frac{d^2 f}{dz^2} - \cancel{\frac{d^2 f}{dz^2}} = 0 \quad \checkmark$$



In fact, any analytic function mapping still satisfies  $\nabla^2 F = 0$   
 $z = (x+iy)^n$

Q: Can we find a complex function that transforms a circle to an airfoil?

Yes  
 and it is simple

$$J(z) = z + \frac{C_1^2}{z}$$

Joukowski Transform

And in fact, there are many discovered mappings functions

Toukowsky (cont)

$$J(z) = z + \frac{c_1^2}{z}$$



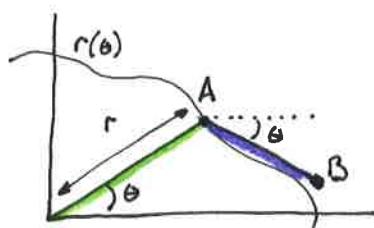
Remember the properties of complex #'s

$$z = x + iy = re^{i\theta} \quad \text{thus} \quad \frac{1}{z} = \frac{1}{|z|} e^{-i\theta} = \frac{1}{r} e^{-i\theta}$$

The map is  $J(z) = z + c_1^2/z$

$$J(z) = r e^{i\theta} + \frac{c_1^2}{r} e^{-i\theta}$$

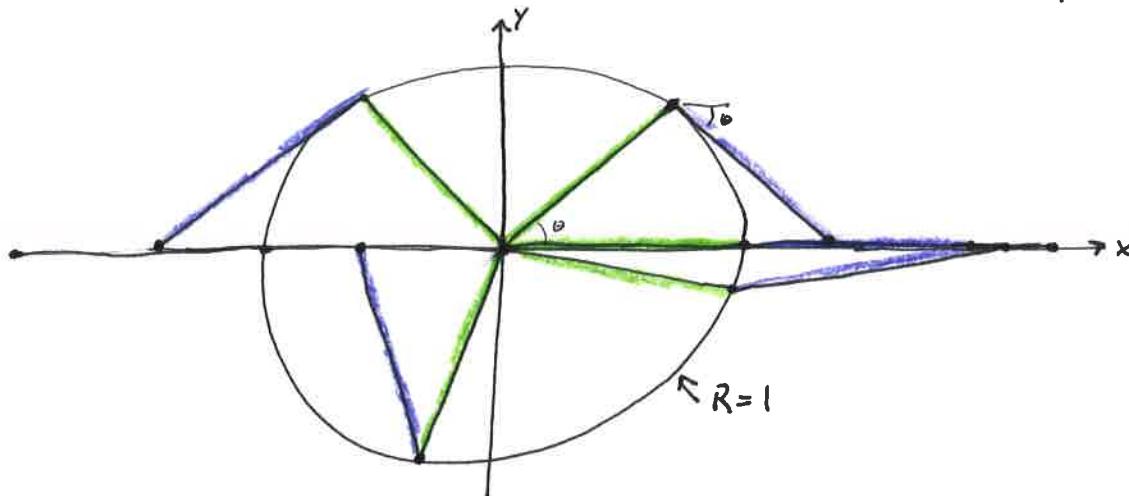
(A)                          (B)



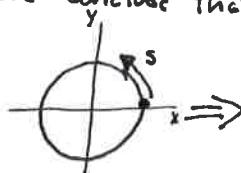
Think of the transform as having two arms or vectors. Arm A traces the unmapped curve (a circle for us). Arm B starts at the end of A, is scaled by  $c_1^2/r$ , and is oriented in the conjugate as A's angle.

Ex:

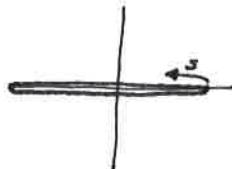
Apply  $J(z) = z + \frac{1}{z}$  to a unit circle at  $x,y = 0,0$        $\frac{c_1^2}{r} = 1$



From this, we conclude that



$z + \frac{1}{z}$  maps a circle at the origin to an infinitesimally thin airfoil with twice the chord.



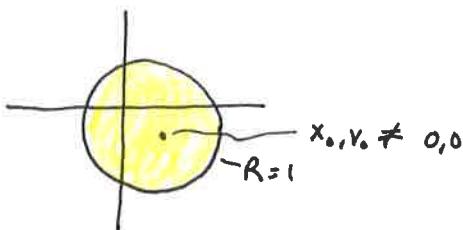
Varying  $C_i^2$  affects the transformed airfoil's thickness (and chord)



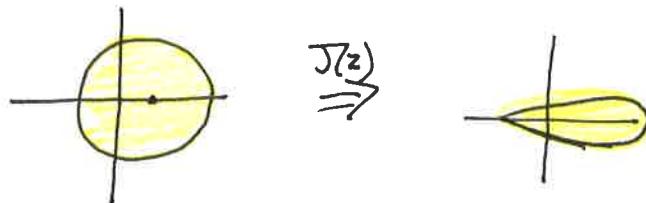
look at  $\theta = 90^\circ$

$$\begin{aligned} \text{circle} &= \uparrow + \downarrow = r_+ - \frac{C_i^2}{r_+} \\ &= \downarrow + \uparrow = r_- + \frac{C_i^2}{r_-} \end{aligned} \Rightarrow \text{thickness} = r_+ - r_- - C_i^2 \left( \frac{1}{r_+} + \frac{1}{r_-} \right)$$

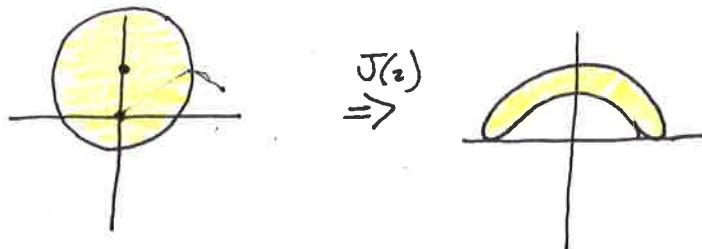
Also, the center of the circle can be offset



- Horizontal offset generates distinctive leading and trailing edges (when  $C_i^2 < 1$ )



- Vertical offset generates camber



- Combinations generate cambered airfoils

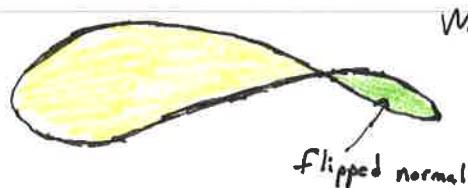


$$x, y = -0.16, 0.23$$

$$C_i = 0.8$$

Q: Could we call this a J162380?

A: I've never heard seen this terminology!



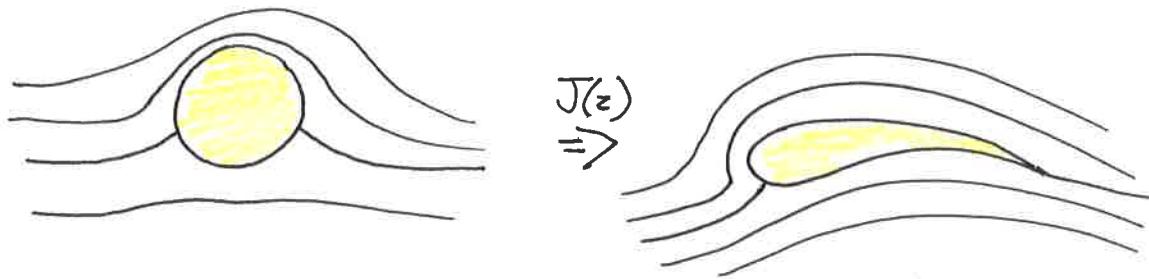
Warning!! Not every combination gives a valid airfoil

$$x, y = -0.32, 0.23$$

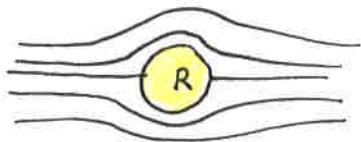
$$C_i = 0.8$$

## Cylinder with Circulation

Now, we have the proper motivation to re-study the ~~circle~~ circle - potential flow.



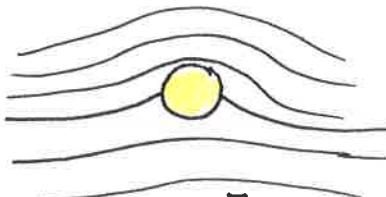
Remember from fluids,



$$\begin{aligned}\Psi &= +V \left( r - \frac{R^2}{r} \right) \sin \theta \\ &= \underbrace{+V r \sin \theta}_{\text{Uniform flow}} + \underbrace{-V \frac{R^2}{r} \sin \theta}_{\text{Doublet}}\end{aligned}$$

→      →      →

Adding a vortex at the center of the circle gives

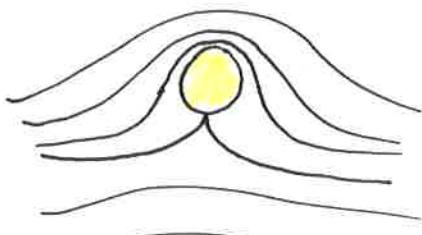


$$\Psi = V r \sin \theta - \frac{V R^2}{r} \sin \theta + \frac{\Gamma}{2\pi r}$$

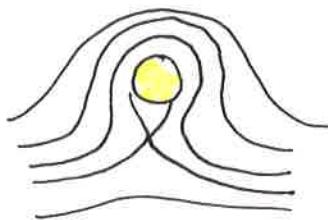
Vortex

Low  $\Gamma$ , 2 stagnation pts.

Remember that adding a vortex is acceptable since the domain we want can be isolated from the vortex with a branch cut.



High  $\Gamma$ , 1 stagnation pt  
( $\Gamma = 4\pi V_a$ )



Super High  $\Gamma$ , no stagnation pt on body.

So, which  $\Gamma$  do we pick?

## Potential Function

- Take velocity potential and stream function and combine into a potential function

$$\omega = \phi + i\psi$$

Warning!!  
W is not  $\nabla \times V$  or  $V_z$

For our cylinder (uniform + doublet + Vortex)<sub>L</sub>

$$\omega = -V \left( z + \frac{R^2}{z} \right) - \frac{i\Gamma}{2\pi} \ln \frac{z}{R}$$

thus  $\phi = \operatorname{Real}(\omega)$   
 $\psi = \operatorname{Imag}(\omega)$

- Write the velocity as a complex #

$$q = u + iv \quad \text{and} \quad \bar{q} = u - iv$$

$$|q| = |u+iv| = |u-iv| = \sqrt{u^2+v^2}$$

From definition of  $\phi$  and  $\psi$ ,

$$u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

Thus

$$\begin{aligned} |q| &= |u+iv| = \left| \frac{\partial \phi}{\partial x} + i(-1) \frac{\partial \psi}{\partial x} \right| \quad \text{or} \\ &= |u-iv| = \left| \frac{\partial \phi}{\partial x} - i(-1) \frac{\partial \psi}{\partial x} \right| = \left| \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} \right| = \left| \frac{\partial}{\partial x} (\phi + i\psi) \right| \\ &= \left| \frac{\partial}{\partial x} (\omega) \right| \quad = \boxed{\left| \frac{\partial \omega}{\partial x} \right| = |q|} \end{aligned}$$

Also

$$|q| = \left| \frac{\partial \omega}{\partial x} \right| = \left| \frac{\partial \omega}{\partial z} \frac{\partial z}{\partial x} \right| \quad \text{but } z = x + iy \Rightarrow \frac{\partial z}{\partial x} = 1$$

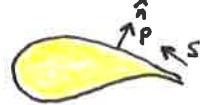
$$\boxed{|q| = \left| \frac{\partial \omega}{\partial z} \right|}$$

# Lift and Moments in a potential flow

Given our mapping  $\circlearrowleft \xrightarrow{J(z)}$ , we need the lift, drag and moment generated.

- One method is to calculate pressures and surface normals, and integrate

$$\text{Force} = \oint_{\text{Airfoil}} P \cdot \hat{n} dS$$



How can we find  $\hat{n}$ ?

- Apply momentum equation in a CV enclosing the airfoil. (FVA eqn 1.28)

$$\underbrace{\iiint \frac{\partial \mathbf{v}}{\partial t} \cdot d\mathbf{v}}_{\text{Steady state}} + \iint P(\mathbf{v} \cdot \mathbf{n}) \mathbf{v} dS = \iiint \rho \mathbf{f} \cdot d\mathbf{v} + \iint -\rho \hat{n} dS + \underbrace{\iint \hat{\tau} \cdot \hat{n} dS}_{\text{No viscosity}} + B$$

Simplify to

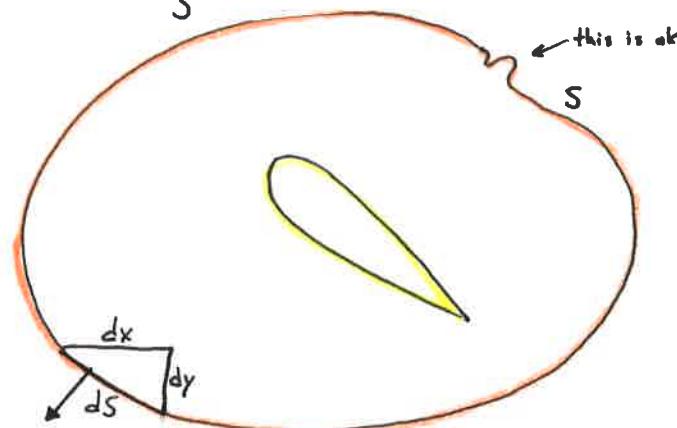
$$\int_S P(u n_x + v n_y + w n_z) \begin{pmatrix} u \\ v \\ w \end{pmatrix} dS = - \int_S \rho \mathbf{a}_x a_y a_z n_x n_y n_z dS + \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{2D}$$

force on airfoil =  $+ B$

Rearrange to solve for  $X, Y, Z$

$$X = + \int_S P n_x dS + \rho \int_S (u n_x + v n_y) v dS$$

$$Y = + \int_S P n_y dS + \rho \int_S (u n_x + v n_y) v dS$$



$$\hat{n} = \begin{pmatrix} -dy \\ dx \\ ds \end{pmatrix}$$

L + M

continued

$$X = - \int_S p dy + \rho \int_S (v dx - u dy) v$$

$$Y = \int_S p dx + \rho \int_S (v dx - u dy) v$$

Compute force in complex frame

$$X - iY = - \int_S p (dy + i dx) + \rho \int_S (v - iv)(v dx - u dy)$$

For low speeds (incomp)

$$P = P_0 - \frac{1}{2} \rho (u^2 + v^2)$$

$$X - iY = - \int_S \left( P_0 - \frac{1}{2} \rho (u^2 + v^2) \right) (dy + i dx) - \rho \int_S (v - iv)(v dy - u dx)$$

Switch signs from above

$$\text{But, } \int_S P_0 dS = 0$$

Thus

$$X - iY = \frac{1}{2} \rho i \int_S \left( \frac{dw}{dz} \right)^2 dz$$

Remember,  $|g| = \left| \frac{dw}{dz} \right|$  is the velocity

Moment:

Similar derivation

$$M = - \frac{1}{2} \rho \operatorname{Real} \left( \int_C \left( \frac{dw}{dz} \right)^2 z dz \right)$$

These are the  
Blasius equations  
developed by .... well....  
Blasius in 1908.