

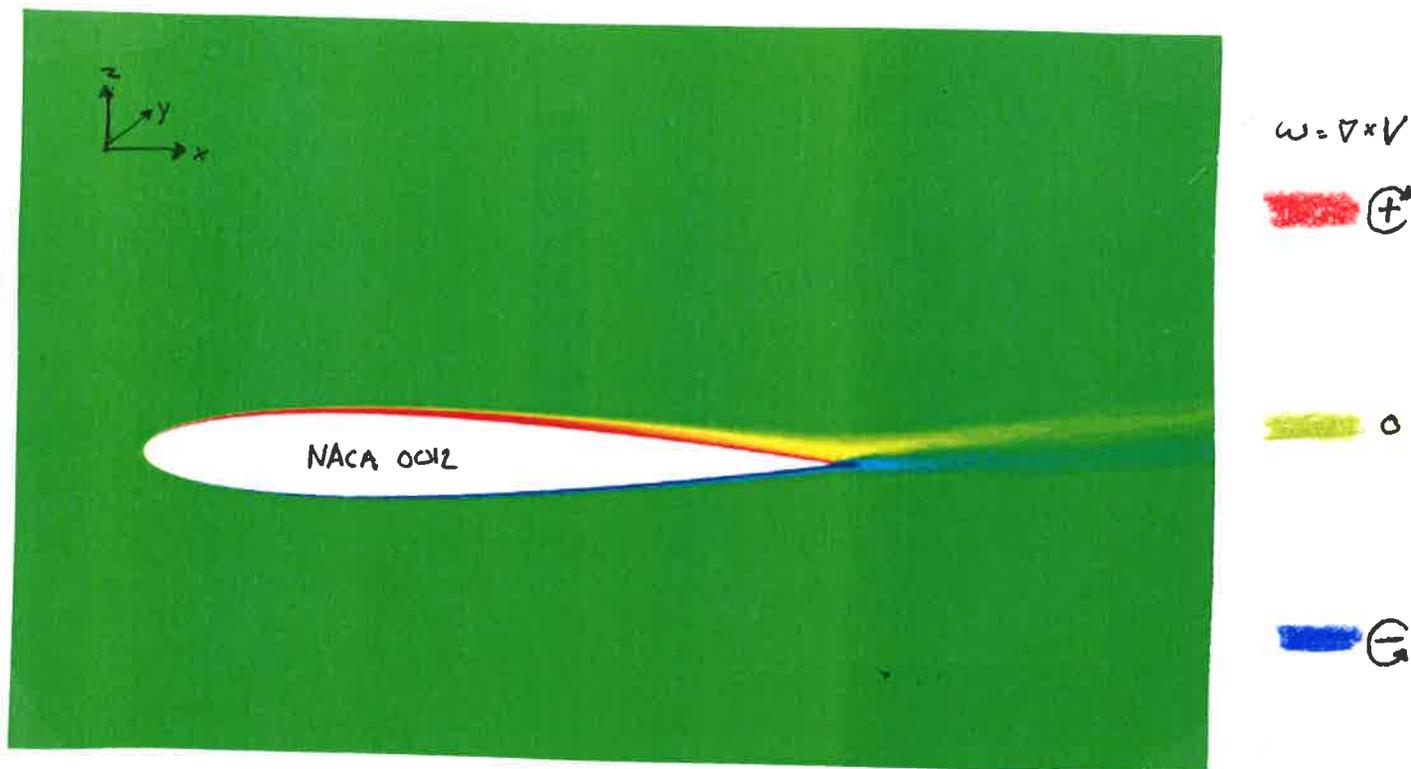
## Lesson 13 (Mod A)

### Thin Airfoil Theory

- preliminaries
- derivation
- Thickness
- Applications

Motivation  
and  
Preliminaries

Vorticity field for NACA 0012 at  $10^\circ$  AOA  $C_l \approx 1.09$



Compute  $\Gamma$ :

$$1) \Gamma = \frac{1}{2} V_\infty c C_l = 1.04$$

2) Integrate  $\omega$

$$\Gamma = \iint_A \omega dA = 1.03$$

3) Contour Integral

$$\Gamma = - \oint \mathbf{V} \cdot d\mathbf{S} \approx 1.01 \quad \text{with } \approx 20 \text{ pts}$$

↑  
Where is vorticity concentrated?

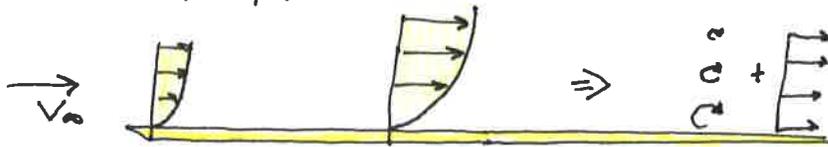
- Near airfoil surface
- Wake

This will be the strategy that we pursue next. Lump all  $\omega$  to the camber line.

What is a vortex? What is the relationship between vorticity (actual) and a vortex (model)?

Vorticity  $\equiv \omega \equiv \nabla \times V$  a measure of <sup>twice</sup> the rotation rate of a fluid particle

Ex: Boundary layer

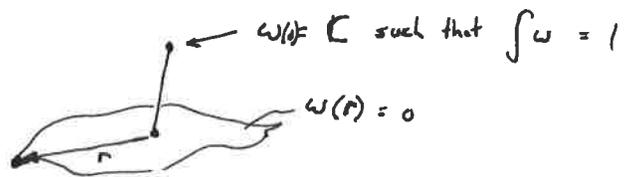


The velocity field consistent (mass and momentum) with the presence of vorticity is

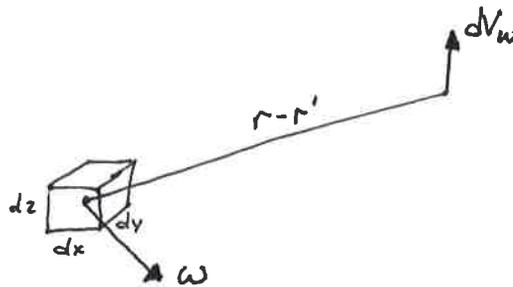
$$V_{\omega}(\vec{r}) = \frac{1}{4\pi} \iiint \omega(r') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} dx' dy' dz'$$

If we take a point source of vorticity

$$V_{\omega} = \frac{1}{4\pi} \int \omega(r') \frac{\vec{r}}{|\vec{r}|^3} dx' dy' dz'$$

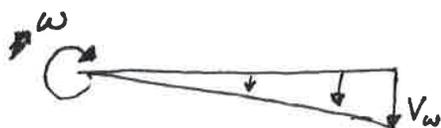


This process is usually too difficult for any practical use other than analytical derivation.



### "Induced Velocity"

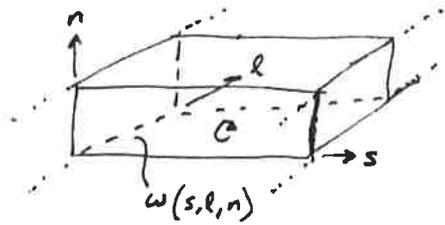
The blob of  $\omega$  at a point is not inducing velocity at a distance.



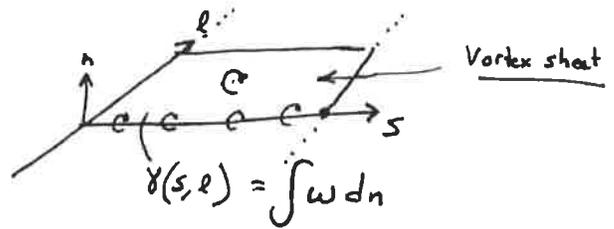
The velocity is just that consistent with conservation of mass and momentum.

There is no spooky action at a distance here!

# Vorticity Lumpins (FVA Fig 2.3)



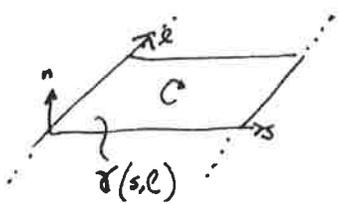
Flatten the n direction  
⇒



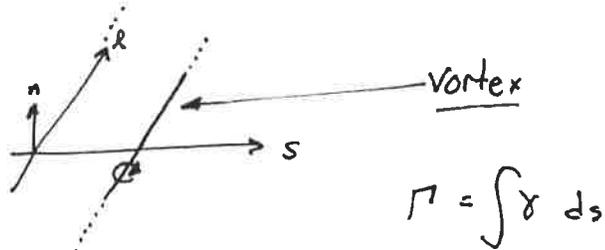
$$V_{\omega} = \frac{1}{4\pi} \iiint \omega(r') \times \frac{r-r'}{|r-r'|^3} dx' dy' dz'$$

$$\Rightarrow V_{\gamma} = \frac{1}{4\pi} \iint \gamma \times \frac{r-r'}{|r-r'|^3} ds dl$$

We compressed all vorticity along an n line (constant s and l) to a <sup>point on a</sup> sheet (s, l).



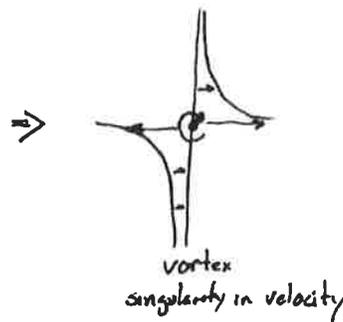
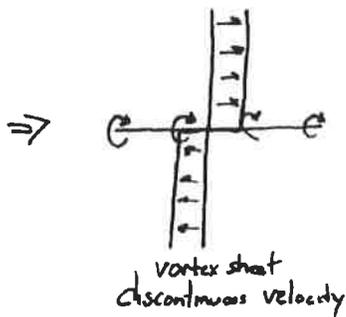
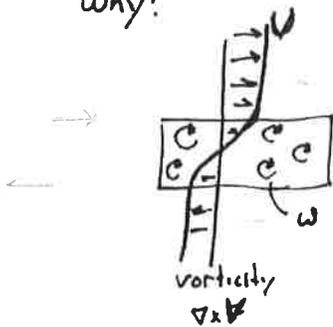
flatten in s direction  
⇒



$$V_{\Gamma} = \frac{1}{4\pi} \int \Gamma \times \frac{r-r'}{|r-r'|^3} dl$$

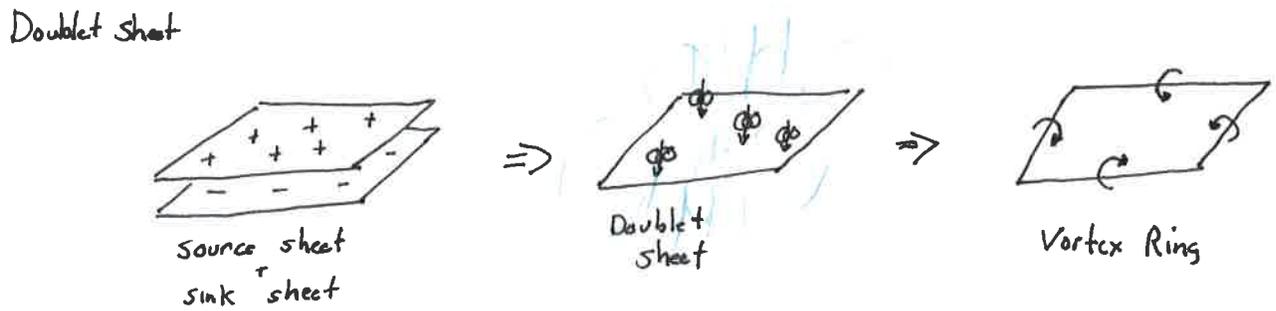
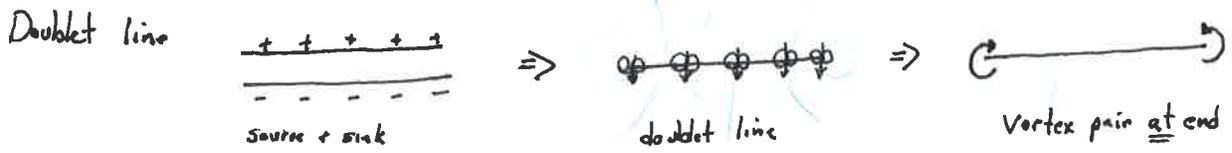
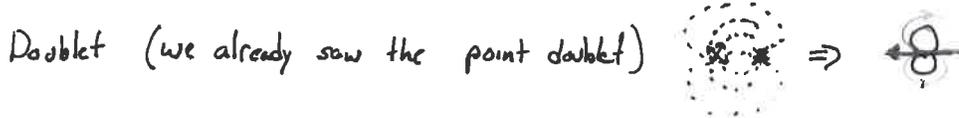
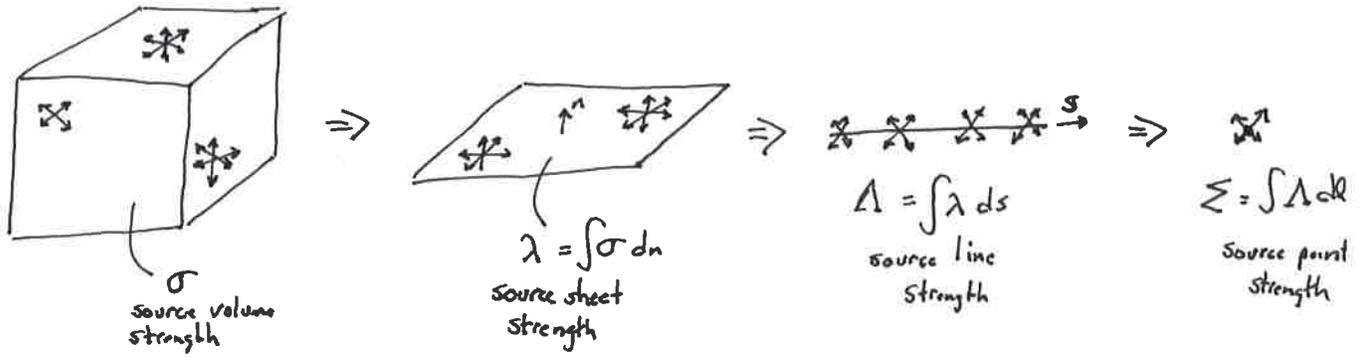
A vortex is a conceptual object generated by lumping vorticity in two directions. Strictly speaking, a vortex is an impossible object (not physically possible).

Why?



$$\nabla \cdot \omega \equiv \nabla \cdot (\nabla \times v) = 0$$

# Lumping Source



These are basic building blocks for 3D aircraft simulation



# Source and Vortex Sheets in 2D and their perturbation velocities

$$\sigma = \nabla \cdot \mathbf{V}$$

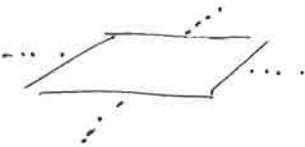
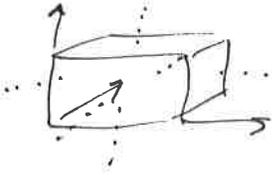
$$\omega = \nabla \times \mathbf{V}$$

reverse

$\Rightarrow$

$$V_{\sigma}(\mathbf{r}) = \frac{1}{4\pi} \iiint \sigma(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dx' dy' dz'$$

$$V_{\omega}(\mathbf{r}) = \frac{1}{4\pi} \iiint \omega(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dx' dy' dz'$$



$\Downarrow$  sheets

$$V_{\sigma}(\mathbf{r}) = \frac{1}{4\pi} \iint \lambda \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} ds dz'$$

$\Downarrow$  2D sheets

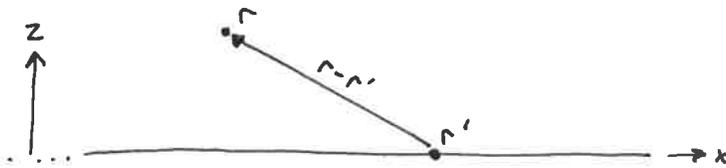
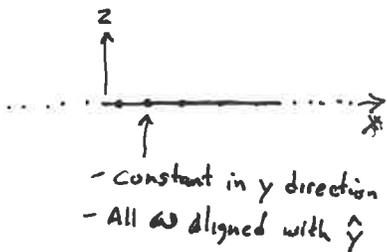
$$V_{\sigma}(\mathbf{r}) = \frac{1}{2\pi} \iint \sigma(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2} dx' dz'$$

$$V_{\omega}(\mathbf{r}) = \frac{1}{2\pi} \iint \omega(\mathbf{r}') \frac{\hat{y} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} dx' dz'$$

Notice

$$|\mathbf{r} - \mathbf{r}'|^2 \text{ not } |\mathbf{r} - \mathbf{r}'|^3$$

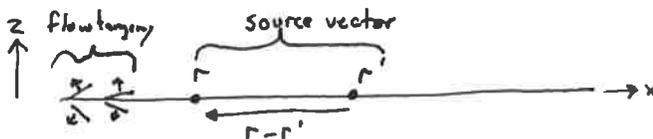
and  $\frac{1}{2\pi}$  vs  $\frac{1}{4\pi}$



$$\mathbf{r} - \mathbf{r}' = (x - x')\hat{x} + (z - z')\hat{z}$$

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{(x - x')^2 + (z - z')^2}$$

Now we see the visual definition of "thin airfoils". All source and vortex properties are determined at and along the x axis, (i.e.  $z - z' = 0$ ). Even flow tangency is applied at  $z = 0$



$$\mathbf{r} - \mathbf{r}' = (x - x')\hat{x}$$

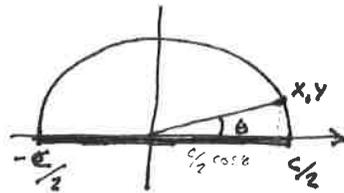
$$|\mathbf{r} - \mathbf{r}'| = \sqrt{(x - x')^2} = |x - x'|$$

$$V_{\sigma}(x) = \frac{1}{2\pi} \int \sigma(x') \frac{(x - x')\hat{x}}{(x - x')^2} dx' = \frac{1}{2\pi} \int \sigma(x') \frac{dx'}{x - x'} \hat{x}$$

$$V_{\omega}(x) = \frac{1}{2\pi} \int \omega(x') \frac{\overbrace{\hat{y} \times (x - x')\hat{x}}^{\hat{y} \times \hat{x} = -\hat{z}}}{(x - x')^2} dx' = \frac{1}{2\pi} \int \omega(x') \frac{-dx'}{x - x'} \hat{z} = V_{\omega}$$

# Thin Airfoil Theory - Part 2

Cosine spacing - a coordinate transform from  $x$  to  $\theta$

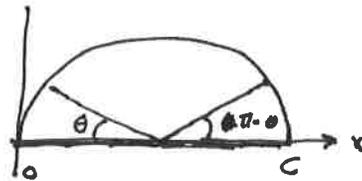


$$x = \frac{c}{2} \cos \theta \Rightarrow dx = -\frac{c}{2} \sin \theta d\theta$$

$$X_{LE} \text{ at } \theta = \pi$$

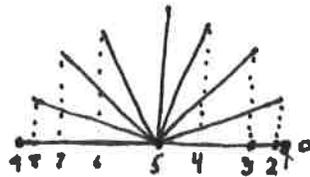
$$X_{TE} \text{ at } \theta = 0$$

or



$$x = \frac{c}{2} (1 - \cos \theta) \Rightarrow dx = \frac{c}{2} \sin \theta d\theta$$

This transform is common in aerodynamics. This is also an excellent way to place points on a line segment when you need more resolution at the ends.



$$8 \text{ steps} \Rightarrow \Delta\theta = \frac{\pi}{8} = 22.5^\circ$$

9 points at

$$0\%, 3.8\%, 15\%, 30\%, 50\%, 70\%, 85\%, 97\%, 100\%$$

There are even an entire class of polynomials that exploit this transform called the Chebyshev polynomials

$$T_n(\cos \theta) = \cos(n\theta)$$

These are orthogonal and satisfy the following ODE

$$(1-x^2)y'' - xy' + n^2y = 0$$

In fact, it has been suggested that if you are not working with harmonic data, the Chebyshev polynomial is ideal (cf. Boyd, Spectral Methods...)

They also have a recurrence equation

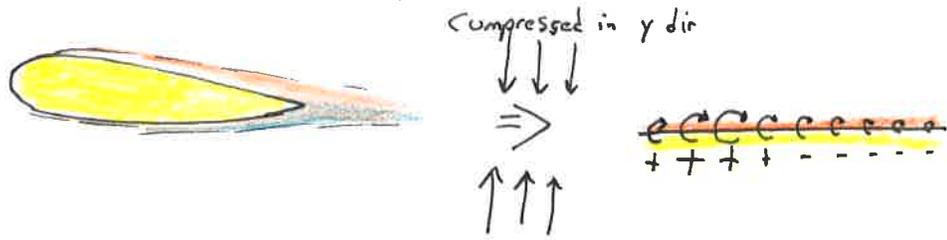
$$T_0(x) = 1, T_1(x) = x, T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

Now, we have the tools  
and theory to derive the

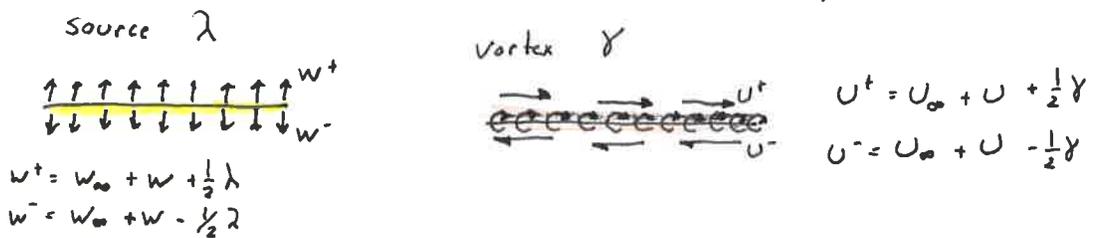
Thin Airfoil Theory

# Conceptual Steps.

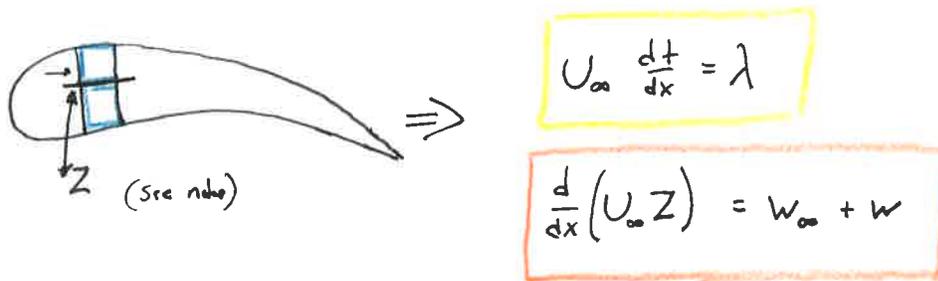
1) Source and vortex terms applied at the x-axis



2) Derive a  $\Delta V$  across the 2D source and vortex sheet (Jump Velocity)



3) With a control volume, establish no flow through the airfoil's top and bottom surface



4) Derive how much the distributed source or vortex sheet affects the perturbation velocities ( $u, w$ )

2D only.

$$u(x) = \frac{1}{2\pi} \int_0^c \lambda(x') \frac{dx'}{x-x'}$$

$$w(x) = \frac{1}{2\pi} \int_0^c -\gamma(x') \frac{dx'}{x-x'}$$

Convert to cosine spacings  $x \rightarrow \theta$

$$u(\theta) = \frac{1}{2\pi} \int_0^\pi \lambda(\theta') \frac{\sin \theta' d\theta'}{\cos(\theta) - \cos \theta'}$$

$$w(\theta) = \frac{1}{2\pi} \int_0^\pi -\gamma(\theta') \frac{\sin \theta' d\theta'}{\cos \theta' - \cos \theta}$$

5) Velocities

$\vec{V}_\infty = \vec{U}_\infty$  same thing

$$W_\infty = V_\infty \sin \alpha$$

$$U_\infty = V_\infty \cos \alpha$$



6) Substitute to give vortex sheet strength equation

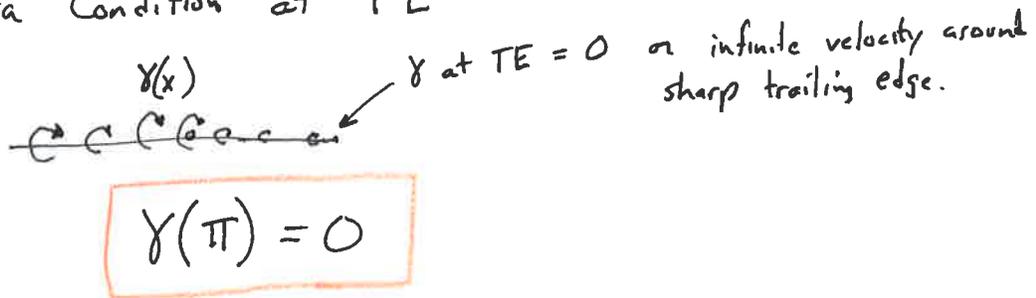
$$\frac{d}{dx}(U_\infty z) = w_\infty + w \Rightarrow U_\infty \frac{dz}{dx}(z) = U_\infty \sin \alpha + \frac{1}{2\pi} \int_0^\pi -\gamma(\theta') \frac{\sin \theta' d\theta'}{\cos \theta' - \cos \theta}$$

$\propto$   
 for small angles

Divide by  $U_\infty$

$$\frac{dz}{dx} = \alpha + \frac{1}{2\pi} \int_0^\pi \frac{-\gamma(\theta')}{U_\infty} \frac{\sin \theta' d\theta'}{\cos \theta' - \cos \theta}$$

7) Apply Kutta Condition at TE



8) Represent  $\alpha - \frac{dz}{dx}$  as a Fourier series

$$A_0 = \alpha - \frac{1}{\pi} \int_0^\pi \frac{dz}{dx} d\theta \quad \text{and} \quad A_n = \frac{2}{\pi} \int_0^\pi \frac{dz}{dx} \cos n\theta d\theta$$

write  $\frac{dz}{dx}$  as a sum of  $\text{---} + \text{~} + \text{~} + \text{~}$

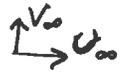
9) Apply Glavert Integral to find solution

$$\frac{\gamma}{U_\infty} = 2 \left( A_0 \frac{1 + \cos \theta}{\sin \theta} + \sum_{n=1}^{\infty} A_n \sin n\theta \right)$$

10) Apply  $\gamma$  to find L, D, M ...

# Extended Thin Airfoil Theory

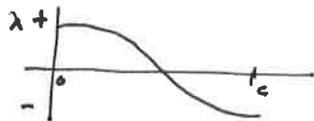
Using a combination of a source and vortex sheet, we can form an inviscid solution to a "thin" airfoil of arbitrary camber and thickness.



To do this, we need a source sheet



where

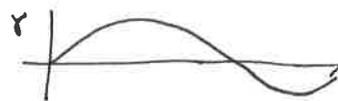


$$\lambda(x) = f(x)$$

and a vortex sheet



where



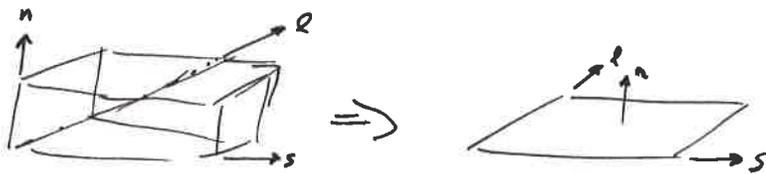
$$\gamma(x) = f(x)$$

such that 1) the flow exactly creates a streamline along the surface  
and

2) The Kutta condition is satisfied.

# Sheet Jumps

Remember, a sheet is a 3D field compressed in one-dimension to give a 2D approximation



$$V_{\sigma} = \frac{1}{4\pi} \iiint \sigma(r') \frac{r-r'}{|r-r'|^3} dx'dy'dz' \Rightarrow V_{\lambda} = \frac{1}{4\pi} \iint \lambda \frac{r-r'}{|r-r'|^3} ds dl$$

$$\lambda = \int \sigma dn$$

Visually we obtain

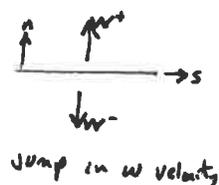
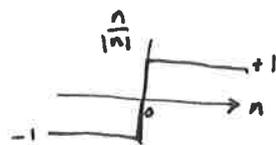


Q: What is the velocity jump from  $0+dn$  to  $0-dn$ ?

$$V_{\lambda}(0,0,n) = \frac{\lambda}{2} \hat{n} \frac{n}{|n|}$$

$\underbrace{\lambda}_{\text{strength}}$      $\underbrace{\hat{n} \frac{n}{|n|}}_{\text{direction}}$

Sign function!



Similarly for a vortex

$$V_{\gamma}(0,0,n) = \frac{\gamma \times n}{2} \frac{n}{|n|}$$

$\underbrace{\gamma \times n}_{\text{Strength and orientation}}$      $\underbrace{\frac{n}{|n|}}_{\text{sign function}}$

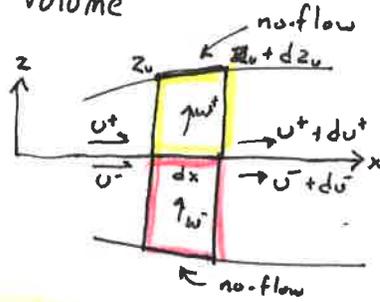
If  $\gamma$  is oriented in the  $l$  direction  $\left( \begin{matrix} \uparrow \\ \otimes \\ \leftarrow \end{matrix} \right) \rightarrow s$  the  $\gamma \times n$  is in the  $s$  direction

Thus, a vortex sheet gives a jump in  $u$  velocity



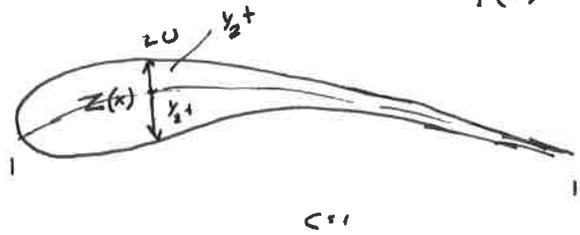
See appendix B in FVA.

# Control Volume



$$z(x)$$

$$+ (k)$$



## Top volume

$$\underbrace{(U^+ + du^+) (z_u + dz_u)}_{\text{out right side}} - \underbrace{U^+ z_u}_{\text{in left}} - \underbrace{w^+ dx}_{\text{in center}} + \underbrace{0}_{\text{out top}} = \underbrace{0}_{\text{steady state}}$$

## Bottom volume

$$\underbrace{(U^- + du^-) (z_e + dz_e)}_{\text{out right}} - \underbrace{U^- z_e}_{\text{in left}} - \underbrace{w^- dx}_{\text{in center}} + \underbrace{0}_{\text{out bottom}} = \underbrace{0}_{\text{steady state}}$$

Rearrange top and divide by dx

$$\frac{1}{dx} \left( \underbrace{U^+ z_u + U^+ dz_u + z_u du^+ + du^+ dz_u}_{\text{out right}} - U^+ z_u \right) - w^+ = 0$$

$$\frac{1}{dx} \left( \underbrace{U^+ dz_u + z_u du^+}_{d(U^+ z_u)} \right) - w^+ = 0 \quad \Rightarrow \quad \frac{d}{dx} (U^+ z_u) - w^+ = 0$$

Apply jump for  $U^+$  and definition of  $z_u = Z + \frac{1}{2} t$

$$\frac{d}{dx} \left( (U_\infty + U + \frac{1}{2} \gamma) (Z + \frac{1}{2} t) \right) - (w_\infty + w + \frac{1}{2} \lambda) = 0$$

similar process for bottom

$$\frac{d}{dx} \left( (U_\infty + U - \frac{1}{2} \gamma) (Z - \frac{1}{2} t) \right) - (w_\infty + w - \frac{1}{2} \lambda) = 0$$

Linear algebra operations (average, subtract)

$$\frac{d}{dx} \left( (U_\infty + U) t + \gamma Z \right) = \lambda$$

$$\frac{d}{dx} \left( (U_\infty + U) Z + \frac{1}{4} \gamma t \right) = w_\infty + w$$

# Traditional "thin airfoil theory" (Glauert)

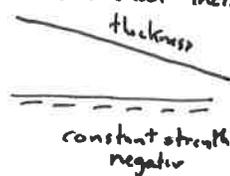
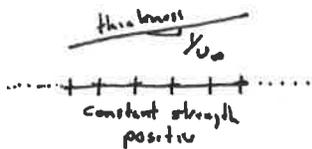
Small angles and small derivatives

Thus,  $\frac{d}{dx} \left( U_\infty t + \underbrace{ut + \delta z}_{\text{higher order terms}} \right) = \gamma \Rightarrow U_\infty \frac{dt}{dx} = \gamma \Rightarrow \boxed{U_\infty \frac{dt}{dx} = \gamma}$

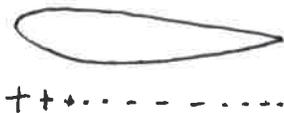
And

$\frac{d}{dx} \left( U_\infty Z + \underbrace{uZ + \frac{1}{4} \delta t}_{\text{again, drop these terms}} \right) = w_\infty + w \Rightarrow \boxed{\frac{d}{dx} (U_\infty Z) = w_\infty + w}$

The first equation indicates that a constant source sheet increases the thickness streamlines by  $\gamma/U_\infty$



To generate an airfoil shape, we would need a strongly positive source tapering to zero at the max thickness followed by a negative source to close out the airfoil.

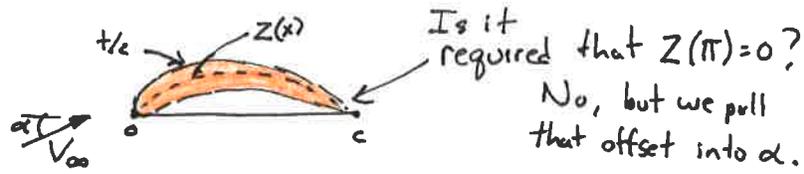


What is the vorticity/vortex strength at the Trailing Edge?

Kutta Condition

$$\boxed{\gamma(TE) = 0}$$

Thin airfoil



$$U_\infty \frac{dt}{dx} = \lambda(x)$$

and

$$\frac{1}{2\pi} \int_0^\pi \frac{\gamma(\theta')}{V_\infty} \frac{\sin \theta' d\theta'}{\cos \theta' - \cos \theta} = \alpha - \frac{dZ(\theta)}{dx} \quad \text{with } \gamma(\pi) = 0$$

Solution:

Represent  $\alpha - \frac{dZ(\theta)}{dx}$  with a Fourier series of cos.

$$\alpha - \frac{dZ(\theta)}{dx} = A_0 - \sum_{n=1}^{\infty} A_n \cos n\theta$$

Why cos? Well, we want  $Z(0) = Z(\pi) = 0$ , so a sine series in  $Z$  is a cosine series in  $\frac{dZ}{dx}$ .

From Fourier theory

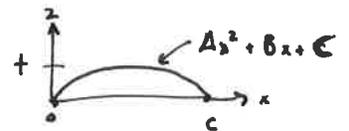
$$A_0 = \alpha - \frac{1}{\pi} \int_0^\pi \frac{dZ}{dx} d\theta \quad \text{and} \quad A_n = \frac{2}{\pi} \int_0^\pi \frac{dZ}{dx} \cos n\theta d\theta$$

If you can write  $\frac{dZ}{dx}$  as a Fourier series,  $Z$  is also a Fourier series.

Remember that

$$\int_0^\pi \cos n\theta \cos m\theta d\theta = \begin{cases} \pi & n=m=0 \\ \pi/2 & n=m \neq 0 \\ 0 & n \neq m \end{cases}$$

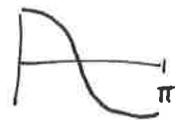
Ex:  $Z(x) = 4\frac{t}{c}x - 4\frac{t}{c^2}x^2$  for a parabolic camberline.



$$\frac{dZ(x)}{dx} = 4\frac{t}{c} - 8\frac{t}{c^2}x$$

convert to cosine transform/space  $x = \frac{c}{2}(1 - \cos \theta)$

$$\frac{dZ(\theta)}{dx} = 4\frac{t}{c} - 8\frac{t}{c^2} \frac{c}{2}(1 - \cos \theta) = 4\frac{t}{c} \cos \theta$$



Match terms

$$A_0 = \alpha - \frac{1}{\pi} \int_0^\pi \dots d\theta \quad A_n = \frac{2}{\pi} \int_0^\pi 4\frac{t}{c} \cos \theta \cos n\theta d\theta$$

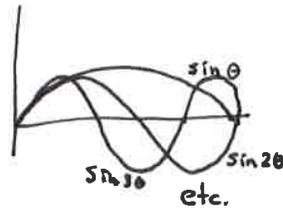
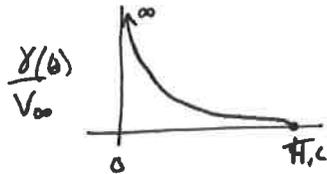
$$A_0 = \alpha, \quad A_1 = 4\frac{t}{c}, \quad A_2 = 0, \quad A_3 = 0, \quad \dots$$

What is  $\gamma(\theta)$ ?

$$\frac{1}{2\pi} \int_0^\pi \frac{\gamma(\theta')}{V_\infty} \frac{\sin \theta' d\theta'}{\cos \theta' - \cos \theta} = A_0 - \sum_{n=1}^{\infty} A_n \cos n\theta$$

$$\frac{\gamma(\theta)}{V_\infty} = 2A_0 \frac{1 + \cos \theta}{\sin \theta} + 2 \sum_{n=1}^{\infty} A_n \sin n\theta$$

Our math elves worked hard to discover this! We won't derive this solution.

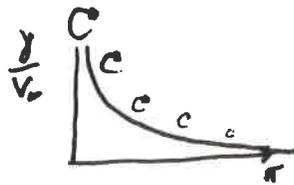


What is  $\lambda(x)$ ?

$$\lambda(x) = V_\infty \frac{dt}{dx}$$

Look at  $A_0$  term

$$\frac{1 + \cos(\theta)}{\sin(\theta)}$$



The vorticity/circulation is stronger at the LE and zero at the TE.

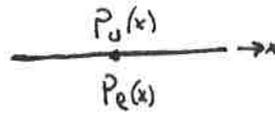
In general, airfoils have rounded LEs and sharp TE's.

N.B

The above integral is called the Glauert Integral

$$\int_0^\pi \frac{\cos n\theta d\theta}{\cos \theta - \cos \phi} = \pi \frac{\sin n\phi}{\sin \phi}$$

# Forces and Moments



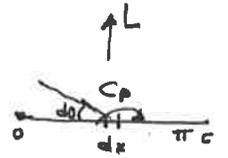
$$\Delta C_p = \frac{P_l - P_u}{\rho} \quad \text{from incompressible B'} \quad \Delta C_p = \frac{|V_u|^2 - |V_l|^2}{V_\infty^2}$$

From the velocity jump from the vortex sheet,

$$\Delta C_p \approx 2 \frac{\gamma}{V_\infty}$$

Lift:

$$C_L = \int_0^c \Delta C_p dx \approx \int_0^c 2 \frac{\gamma}{V_\infty} dx = \int_0^\pi \underbrace{\frac{\gamma}{V_\infty} \sin \theta d\theta}_{\text{cosine transform.}}$$



$$= \int_0^\pi 2 \left( A_0 \frac{1 + \cos \theta}{\sin \theta} + \sum_{n=1}^{\infty} A_n \sin n\theta \right) \sin \theta d\theta$$

with  $\int_0^\pi \sin \theta \cos \theta d\theta$  and  $\int_0^\pi \sin n\theta \sin m\theta d\theta \neq 0$  only when  $n = m$   
 $= \frac{\pi}{2}$  when  $n = m$

$$C_L = 2\pi A_0 + \cancel{2\pi} \pi A_1$$

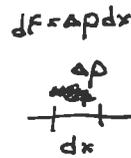
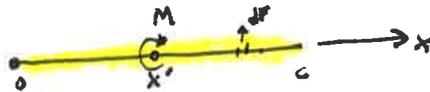


$$= 2\pi \alpha + 2 \int_0^\pi \frac{dz}{dx} (\cos \theta - 1) d\theta$$

$$= 2\pi \alpha + C_{L_0} \quad \text{where } C_{L_0} \text{ depends on the camberline}$$

$$\boxed{\frac{dC_L}{d\alpha} = 2\pi}$$

# Moment



Cosine expansion,

$$x = \frac{c}{2}(1 - \cos \theta)$$

$$dx = \frac{c}{2} \sin \theta d\theta$$

$$\frac{\Delta P}{g} = \Delta C_p$$

Compute Moment about  $x'$

$$C_{m_{x'}} = \frac{M_{x'}}{g c^2} = \frac{-\int_0^c (x-x') dF}{g c^2} = \frac{-\int_0^c (x-x') \Delta P dx}{g c^2} = \frac{-\int_0^c (x-x') \Delta C_p dx}{c^2}$$

$$= -\int_0^\pi \left( \frac{c}{2}(1 - \cos \theta) - \frac{c}{2}(1 - \cos \theta') \right) \Delta C_p \frac{c}{2} \sin \theta d\theta \cdot \frac{1}{c^2}$$

$$= -\int_0^\pi \frac{1}{2} (1 - \cos \theta - 1 + \cos \theta') \Delta C_p \frac{1}{2} \sin \theta d\theta$$

$$= -\int_0^\pi \frac{1}{4} (\cos \theta' - \cos \theta) \sin \theta \left( 2 A_0 \frac{1 + \cos \theta}{\sin \theta} + 2 \sum_{n=1}^{\infty} A_n \sin n\theta \right) d\theta$$

$$= -\frac{1}{2} \int_0^\pi (\cos \theta' - \cos \theta) A_0 (1 + \cos \theta) d\theta - \frac{1}{2} \int_0^\pi (\cos \theta' - \cos \theta) \sum_{n=1}^{\infty} A_n \sin n\theta d\theta$$

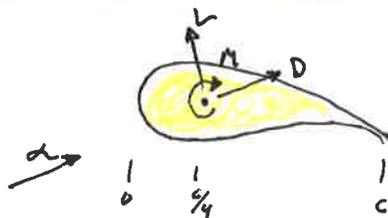
Orthogonal terms ..... Some algebra.

$$= \underbrace{-\frac{1}{2} A_0 \pi \cos \theta' + \frac{\pi}{4} A_0}_{\text{depends on } \alpha} - \underbrace{\frac{1}{2} A_1 \frac{\pi}{2} \cos \theta' + \frac{1}{2} A_2 \frac{\pi}{4}}_{\text{does not depend on } \alpha}$$

Remove  $\alpha$  dependency by  $-\frac{1}{2} A_0 \pi \cos \theta' + \frac{\pi}{4} A_0 = 0$  !!!

$$\cos \theta' = \frac{1}{2} \Rightarrow x' = \frac{c}{2} \left( 1 - \frac{1}{2} \right) = \frac{c}{4}$$

The quarter chord moment is independent of  $\alpha$ .



Now, we can look at  
thickness

# Extended Thin Airfoil Theory

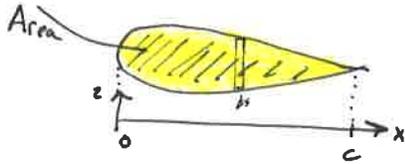
Now, consider the thickness.  $\lambda(x) = \frac{d}{dx}((U_\infty + u)t + \gamma Z) \approx V_\infty \frac{dt}{dx}$

Write thickness as Fourier sine series. Why sine? Zero at endpoints (0, c)

$$t(\theta) = c \sum_{n=1}^{\infty} B_n \sin n\theta \quad \text{---} + \text{---} + \text{---} + \dots$$

$$B_n = \frac{2}{\pi} \int_0^\pi \frac{t}{c} \sin n\theta d\theta$$

Airfoil Area



$$\text{Area} = \int_0^c t dx = \frac{c}{2} \int_0^\pi t \sin \theta d\theta$$

orthogonal, so only  $B_1$  term contributes

$$= \frac{\pi}{4} c^2 B_1$$

$$\Rightarrow B_1 = \frac{4A}{\pi c^2}$$

Source Sheet

$$\lambda(\theta) = V_\infty \frac{dt(\theta)}{dx} = V_\infty \frac{dt}{d\theta} \frac{d\theta}{dx} = \dots$$

$$x = \frac{c}{2}(1 - \cos \theta) \Rightarrow \frac{dx}{d\theta} = \frac{c}{2} \sin \theta$$

$$t = c \sum B_n \sin n\theta \Rightarrow \frac{dt}{d\theta} = c \sum n B_n \cos n\theta$$

$$\lambda(\theta) = V_\infty \cdot \frac{2}{c \sin \theta} \cdot c \sum n B_n \cos n\theta = V_\infty \frac{2}{\sin \theta} \sum_{n=1}^{\infty} n B_n \cos n\theta$$

# Velocity Perturbation

Glauert Integral  $\int_0^\pi \frac{\cos n\theta' d\theta'}{\cos\theta' - \cos\theta} = \pi \frac{\sin n\theta}{\sin\theta}$

$$u(x) = \frac{1}{2\pi} \int_0^c \lambda(x') \frac{dx'}{x-x'}$$

$$= \frac{1}{2\pi} \int_0^\pi V_\infty \frac{2}{\sin\theta'} \left[ \sum_{n=1}^\infty n B_n \cos n\theta' \right] \cdot \frac{c}{2} \sin\theta' d\theta'$$

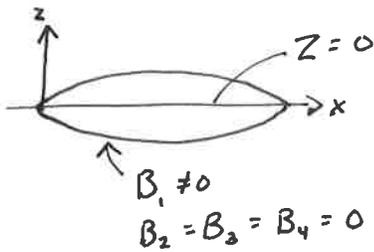
$$x' = \frac{c}{2} (1 - \cos\theta')$$

$$dx' = \frac{c}{2} \sin\theta' d\theta'$$

$$\frac{c}{2} (1 - \cos\theta') - \frac{c}{2} (1 - \cos\theta)$$

$$u(\theta) = V_\infty \sum_{n=1}^\infty n B_n \frac{\sin n\theta}{\sin\theta}$$

# Flat elliptical thickness airfoil



$$\frac{d}{dx} \left( (U_\infty + u) t + \gamma z \right) = \lambda$$

$$\frac{d}{dx} \left( (U_\infty + u) z + \frac{1}{4} \gamma t \right) = w_\infty + w$$

thus

$$\frac{1}{4} \frac{d}{dx} (\gamma t) = w_\infty + w$$

with

$$x = \frac{c}{2} (1 - \cos\theta) \quad (\text{cosine transform})$$

and

$$t = c B_1 \sin\theta \quad (\text{just 1 term})$$

$$\text{and } \frac{d}{dx} = \frac{d}{d\theta} \frac{d\theta}{dx}$$

plug into  $\frac{1}{4} \frac{d}{dx} (\gamma t) = w_\infty + w$

$$\frac{1}{4} \frac{2}{c \sin\theta} \frac{d}{d\theta} \left( \underbrace{\gamma(\theta) c B_1 \sin\theta}_{\text{differential eq.}} \right) = V_\infty \sin\theta + \frac{1}{2\pi} \int_0^\pi \underbrace{-\gamma(\theta') \frac{\sin\theta' d\theta'}{\cos\theta' - \cos\theta}}_{\text{integral eq.}}$$

Integro-Differential Equ. for  $\gamma(\theta)$

Solution!!

$$\frac{\gamma}{V_\infty} = \frac{2\alpha}{1-B_1} \frac{1+\cos\theta}{\sin\theta}$$

Same as flat airfoil mult' by  $\frac{1}{1-B_1}$

# Forces and Moments

We already did this with  $\frac{\gamma}{V_\infty} = 2\alpha \frac{1+\cos\theta}{\sin\theta}$  !



$$\Delta C_p = \frac{P_l - P_u}{\rho} \approx \frac{2\gamma}{V_\infty}$$

Lift

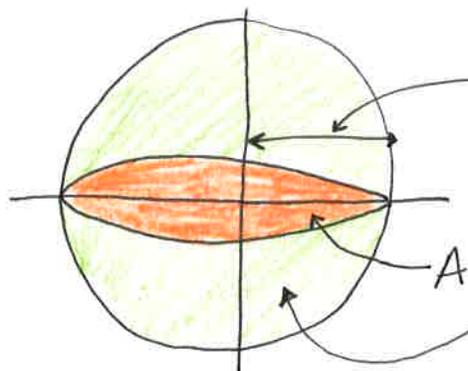
$$C_L = \int_0^c \Delta C_p dx \approx \int_0^c 2 \frac{\gamma}{V_\infty} dx = \int_0^\pi \frac{\gamma}{V_\infty} \sin\theta d\theta$$

$$= \int_0^\pi \frac{2\alpha}{1-B_1} \frac{1+\cos\theta}{\sin\theta} \sin\theta d\theta = \int_0^\pi \frac{2\alpha}{1-B_1} (1+\cos\theta) d\theta$$

$$= \frac{2\alpha}{1-B_1} \Big|_0^\pi = \frac{2\pi\alpha}{1-B_1} = \left(2\pi\right) \left(\frac{1}{1-4A} \frac{1}{\pi c^2}\right) \alpha$$

$$C_{L\alpha} = \frac{2\pi}{1 - \frac{4A}{\pi c^2}}$$

Geometric interpretation of  $\frac{1}{1 - \frac{4A}{\pi c^2}}$



$\frac{c}{2}$  is half chord

$\pi \left(\frac{c}{2}\right)^2$  is the area of a circle with points at the LE and TE

Thus  $\frac{4A}{\pi c^2}$  is the area ratio of the airfoil and circle.

Does  $C_{L\alpha}$  approach  $\infty$  as the airfoil thickens to a circle? No. This result is only good for thin airfoils.

$\frac{1}{1-A/R}$  = inverse of 1 - area ratio

No surprise that the ref Area is a circle since we assumed only  $B_1 \neq 0$  for an elliptical thickness.

# Moment

$$\begin{aligned}
 C_{m_{1/4}} &= \frac{M}{\rho c^2} = \frac{1}{\rho c^2} \int_0^c -\Delta C_p \left(x - \frac{c}{4}\right) dx = \frac{1}{\rho c^2} \int_0^c -\frac{2\gamma}{V_\infty} \left(x - \frac{c}{4}\right) dx \\
 &= \frac{1}{\rho c^2} \int_0^\pi -\frac{\gamma}{V_\infty} \left(\frac{c}{2}(1 - \cos\theta) - \frac{c}{4}\right) \sin\theta d\theta \\
 &= \frac{1}{\rho c^2} \int_0^\pi -\left(\frac{2\alpha}{1-\beta_1}\right) \left(\frac{1 + \cos\theta}{\sin\theta}\right) \left(\frac{c}{2}(1 - \cos\theta) - \frac{c}{4}\right) \sin\theta d\theta \\
 &= \frac{1}{\rho c^2} \int_0^\pi \left(-\frac{2\alpha}{1-\beta_1}\right) (1 + \cos\theta)(c) \left(\frac{1}{4} - \frac{\cos\theta}{2}\right) d\theta \\
 &= 0 \quad !!!
 \end{aligned}$$

Where is the a.c. shift?

What is the mathematical reason that the integral is zero?

Compare with the cambered thin airfoil

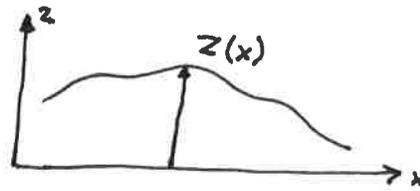
$$\begin{aligned}
 C_{m_{1/4}} &= \frac{1}{\rho c^2} \int_0^\pi -\left(2A_0 \frac{1 + \cos\theta}{\sin\theta} + 2A_1 \sin\theta + 2A_2 \sin 2\theta + \dots\right) \left(\frac{c}{2}(1 - \cos\theta) - \frac{c}{4}\right) \sin\theta d\theta \\
 &= -\frac{\pi}{4} (A_1 - A_2) \quad \text{where } \overset{\text{only}}{\checkmark} A_0 \text{ has an } \alpha \text{ term}
 \end{aligned}$$

$$\frac{dC_{m_{1/4}}}{d\alpha} = 0$$

Applications

# Summary of thin airfoil theory (i.e. How to apply)

- Determine mean camber line  $Z(x)$



- Convert  $Z(x)$  to cosine transform spacing

$$x = \frac{c}{2} (1 - \cos \theta)$$

- Compute Fourier terms

$$A_0 = \alpha - \frac{1}{\pi} \int_0^{\pi} \frac{dZ}{dx} d\theta$$

$$A_n = \frac{2}{\pi} \int_0^{\pi} \frac{dZ}{dx} \cos n\theta d\theta$$

where  $\frac{dZ(\theta)}{dx}$  is a function of  $\theta$  not  $x$ .

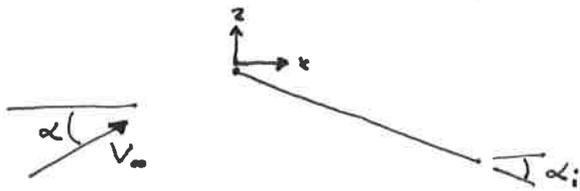
- Compute aero terms

$$C_l = 2\pi A_0 + \pi A_1$$

$$C_{m/c_4} = -\pi \frac{1}{4} (A_1 - A_2)$$

$$C_d = 0$$

# Flat Airfoil (with incidence angle)



$\alpha_i$  is the incidence angle.

Obviously, the easy way to solve this problem is to add  $\alpha + \alpha_i$  for the effective AOA. Here we do the analysis the hard way!

- Mean Camber line

$$Z(x) = -\tan(\alpha_i) \cdot x$$

- Convert to CS

$$Z(\theta) = -\tan(\alpha_i) \cdot \underbrace{\frac{c}{2}}_x (1 - \cos \theta)$$

- Calculate  $\frac{dz}{dx}$

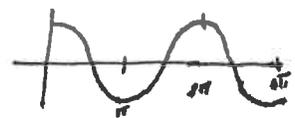
$$\frac{dz}{dx} = -\tan(\alpha_i)$$

- Fourier terms

$$\begin{aligned} A_0 &= \alpha - \frac{1}{\pi} \int_0^\pi -\tan(\alpha_i) d\theta = \alpha + \frac{\tan(\alpha_i)}{\pi} \int_0^\pi d\theta \\ &= \alpha + \tan(\alpha_i) \approx \alpha + \alpha_i \quad \text{when } \alpha_i \text{ is small} \end{aligned}$$

$$\begin{aligned} A_n &= \frac{2}{\pi} \int_0^\pi -\tan(\alpha_i) \cos n\theta d\theta \\ &= \underbrace{-\frac{2}{\pi} \int_0^\pi \cos(n\theta) d\theta}_0 \cdot \tan(\alpha_i) = 0 \end{aligned}$$

0 always

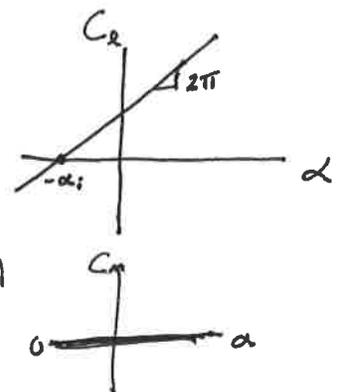


- Compute aero terms

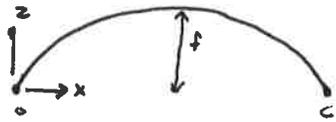
$$C_L = 2\pi A_0 + \pi A_1 = 2\pi(\alpha + \alpha_i)$$

$$C_{L_{\alpha_i}} = 2\pi \quad C_{L_0} = 2\pi \alpha_i$$

$$C_{m_{c/4}} = -\frac{\pi \rho V_\infty^2 c^2}{4} (A_1 - A_2) = 0 \quad \text{symmetrical!}$$



# Circular Arc Airfoil



pick a parabolic shape  $Z(x) = Ax + Bx^2$  such that  $Z(0) = Z(c) = 0$   
and  $Z(\frac{c}{2}) = f$

$$Z(x) = \frac{4f}{c^2}x - \frac{4f}{c^2}x^2$$

Calculate  $\frac{dz}{dx}$

$$x = \frac{c}{2}(1 - \cos \theta)$$

$$\begin{aligned} \frac{dz}{dx} &= \frac{4f}{c^2} - \frac{8f}{c^2}x = \frac{4f}{c^2} - \frac{4f}{c^2}c(1 - \cos \theta) \\ &= \frac{4f}{c^2} - \frac{4f}{c^2} + \frac{4f}{c^2} \cos \theta = \underbrace{\frac{4f}{c^2} \cos \theta}_{\text{convenient, huh?!}} \end{aligned}$$

Fourier terms

$$A_0 = \alpha - \frac{1}{\pi} \int_0^\pi \underbrace{\frac{4f}{c^2} \cos \theta}_0 d\theta = \alpha$$

$$A_n = \frac{2}{\pi} \int_0^\pi \underbrace{\frac{4f}{c^2} \cos \theta}_{\frac{dz}{dx}} \cos n\theta d\theta \quad \text{only has value when } n=1$$

$$A_1 = \frac{4f}{c^2} \quad A_2 = A_3 = A_4 = A_{\dots} = 0$$

Compute Aero terms

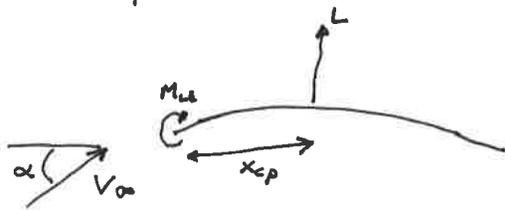
$$C_L = 2\pi A_0 + \pi A_1 = 2\pi \alpha + \pi \frac{4f}{c^2} = 2\pi \left( \alpha + \frac{2f}{c} \right) = C_L$$

$$C_{m/c_4} = -\pi \frac{1}{4} (A_1 - A_2) = -\pi \frac{1}{4} \left( \frac{4f}{c^2} - 0 \right) = -\pi f/c = C_{m/c_4}$$

Compare with Joukowski:

Exactly the same!

# Center of Pressure



The location of the center of pressure ( $x_{cp}$ ) is where the moment is zero

$$x_{cp} = \frac{-M_{LE}}{L} = \frac{-M_{LE}}{L} \cdot \frac{8c^2}{8c^2} = -\frac{C_{m_{LE}} \cdot c}{C_L}$$

$$C_{m_{LE}} = -\left(\frac{C_{L_0}}{4} + \frac{\pi}{4}(A_1 - A_2)\right)$$

$$x_{cp} = \frac{\frac{C_{L_0}}{4} + \frac{\pi}{4}(A_1 - A_2)}{C_L} \cdot c = \frac{c}{4} \left(1 + \frac{\pi}{C_L} (A_1 - A_2)\right)$$

write in terms of  $C_{m_{\frac{1}{4}}}$   
 $A_2 - A_1 = \frac{4}{\pi} C_{m_{\frac{1}{4}}}$

$$= \frac{c}{4} \left(1 + \frac{-4C_{m_{\frac{1}{4}}}}{C_L}\right)$$

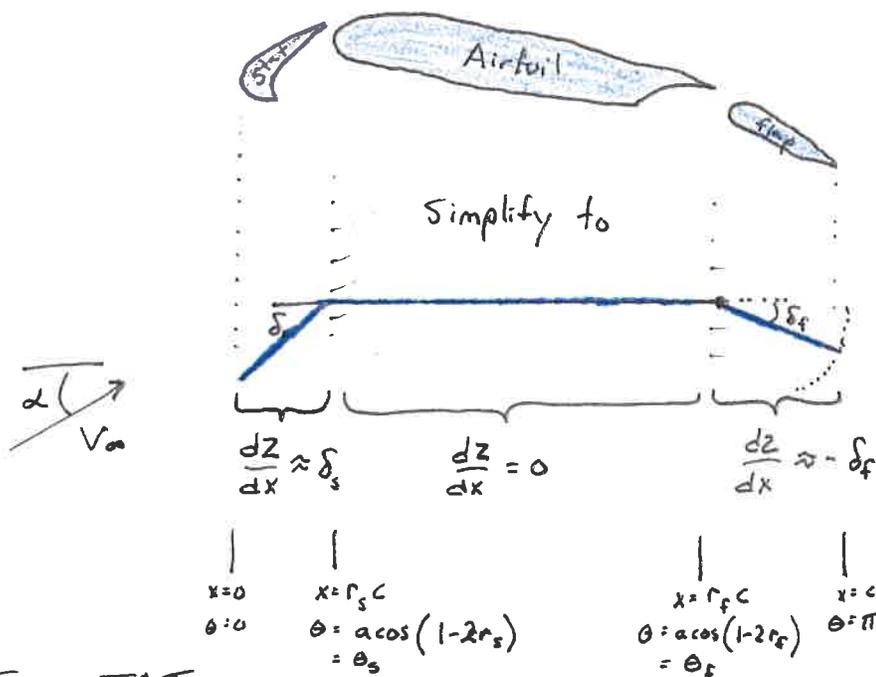
$C_{m_{\frac{1}{4}}}$  is constant for a given airfoil, but  $C_L$  depends on  $\alpha$

$$x_{cp} = \frac{c}{4} \left(1 - \frac{4C_{m_{\frac{1}{4}}}}{2\pi\alpha + C_{L_0}}\right)$$

The center of pressure moves as AOA and camber change. ~~At~~ <sup>At</sup> zero lift, the  $x_{cp} = \frac{c}{4} \left(1 - \frac{4C_{m_{\frac{1}{4}}}}{0}\right) \rightarrow \infty$  Not even on the airfoil!!

Using  $x_{cp}$  is strongly discouraged, use  $x_{ac}$  instead.

# Airfoil with slats + Flaps



$$x = \frac{c}{2} (1 - \cos \theta)$$

$$\theta = \alpha \cos \left( 1 - 2 \frac{x}{c} \right)$$

precisely;

$$\frac{dz}{dx} = \tan \delta$$

From TAT

$$\begin{aligned}
 A_0 &= \alpha - \frac{1}{\pi} \int_0^\pi \frac{dz(\theta)}{dx} d\theta = \alpha - \frac{1}{\pi} \int_0^{\theta_s} \delta_s d\theta - \frac{1}{\pi} \int_{\theta_s}^{\theta_f} 0 d\theta - \frac{1}{\pi} \int_{\theta_f}^\pi -\delta_f d\theta \\
 &= \alpha - \frac{\delta_s}{\pi} (\theta_s - 0) - \frac{1}{\pi} (0) + \frac{\delta_f}{\pi} (\pi - \theta_f) \\
 &= \alpha - \frac{\delta_s \theta_s}{\pi} - \frac{\delta_f \theta_f}{\pi} + \frac{\delta_f \pi}{\pi} = \alpha - \frac{1}{\pi} (\delta_s \theta_s + \delta_f \theta_f) + \delta_f
 \end{aligned}$$

$$\begin{aligned}
 A_n &= \frac{2}{\pi} \int_0^\pi \frac{dz}{dx} \cos n\theta d\theta = \frac{2}{\pi} \int_0^{\theta_s} \delta_s \cos n\theta d\theta + \frac{2}{\pi} \int_{\theta_s}^{\theta_f} 0 \cos n\theta d\theta + \frac{2}{\pi} \int_{\theta_f}^\pi -\delta_f \cos n\theta d\theta \\
 &= \frac{2}{\pi} \delta_s \int_0^{\theta_s} \cos n\theta d\theta + \frac{2}{\pi} 0 + \frac{-2}{\pi} \delta_f \int_{\theta_f}^\pi \cos n\theta d\theta \\
 &= \frac{2}{\pi} \delta_s \frac{1}{n} \sin n\theta_s - \frac{2}{\pi} \delta_f \frac{1}{n} \sin n\pi + \frac{2}{\pi} \delta_f \frac{1}{n} \sin n\theta_f = \frac{2}{n\pi} (\delta_s \sin n\theta_s + \delta_f \sin n\theta)
 \end{aligned}$$

$$A_1 = \frac{2}{\pi} (\delta_s \sin \theta_s + \delta_f \sin \theta_f)$$

$$= \frac{2}{\pi} (\delta_s \sqrt{4r_s + 4r_s^2} + \delta_f \sqrt{4r_f + 4r_f^2})$$

$$\sin(\alpha \cos(x)) = \sqrt{1-x^2}$$

$$\theta_s = \alpha \cos(1-2r_s) \Rightarrow \sin \theta_s = \sqrt{4r_s + 4r_s^2}$$

$$C_e = 2\pi A_0 + \pi A_1$$

$$= 2\pi \left( \alpha - \frac{1}{\pi} (\delta_s \theta_s + \delta_f \theta_f) + \delta_f \right) + \pi \frac{2}{\pi} (\delta_s \sin \theta_s + \delta_f \sin \theta_f)$$

$$= 2\pi \alpha + 2\pi \left( \delta_f - \frac{\delta_f \theta_f}{\pi} + \frac{\delta_f \sin \theta_f}{\pi} \right) + 2\pi \left( -\frac{\delta_s \theta_s}{\pi} + \frac{\delta_s \sin \theta_s}{\pi} \right)$$

$$= \underbrace{2\pi \alpha}_{\text{AOA}} + \underbrace{2\pi \delta_f \left( 1 - \frac{\theta_f}{\pi} + \frac{\sin \theta_f}{\pi} \right)}_{\text{Flap}} + \underbrace{2\pi \delta_s \left( -\frac{\theta_s}{\pi} + \frac{\sin \theta_s}{\pi} \right)}_{\text{Slat}}$$

Notice the difference between a flap  $\left( 1 - \frac{\theta_f}{\pi} + \frac{\sin \theta_f}{\pi} \right)$  and a slat  $\left( -\frac{\theta_s}{\pi} + \frac{\sin \theta_s}{\pi} \right)$ .

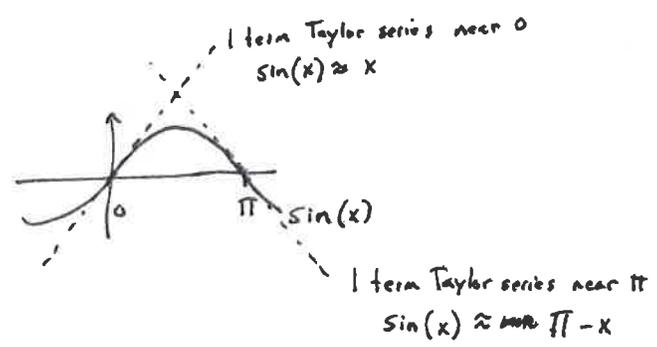
Why? The flap directly affects/contributes to  $C_e$  with  $\delta_f$ , but not for the slat!!

Flaps have the Kutta condition applied. Slats do not.



- Limiting case when slat and flap are small
  - Zero order
    - $\sin \theta_s \approx \theta_s$  when  $\theta_s$  is near 0.
    - $\sin(\theta_f) \approx \theta_f$  when  $\theta_f$  is near  $\pi$

$$C_e = 2\pi \alpha + 2\pi \delta_f$$



- 1st order
  - $\sin \theta_s \approx \theta_s$
  - $\sin \theta_f \approx \pi - \theta_f$  near  $\pi$

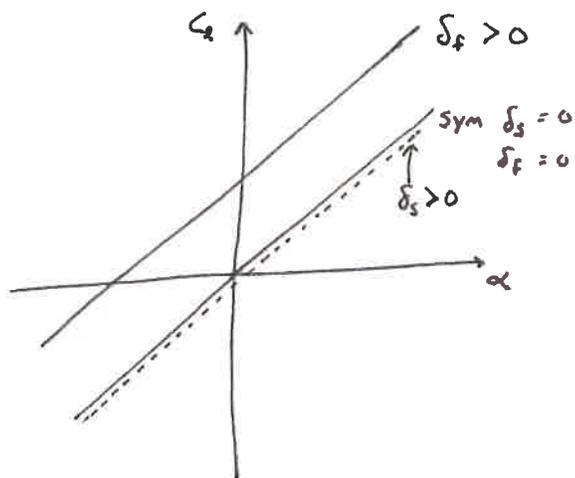
$$C_e = 2\pi \alpha + 2\pi \delta_f \left( 1 - \frac{\theta_f}{\pi} + \frac{\pi}{\pi} - \frac{\theta_f}{\pi} \right) + 2\pi \delta_s \left( -\frac{\theta_s}{\pi} + \frac{\theta_s}{\pi} \right)$$

$$C_e = \underbrace{2\pi \alpha}_{\text{AOA}} + \underbrace{4\pi \delta_f}_{\text{!!!!}} - \cancel{4\theta_f \delta_f} + 0$$

Flaps are efficient at increasing lift at a given AOA. Slats are not.

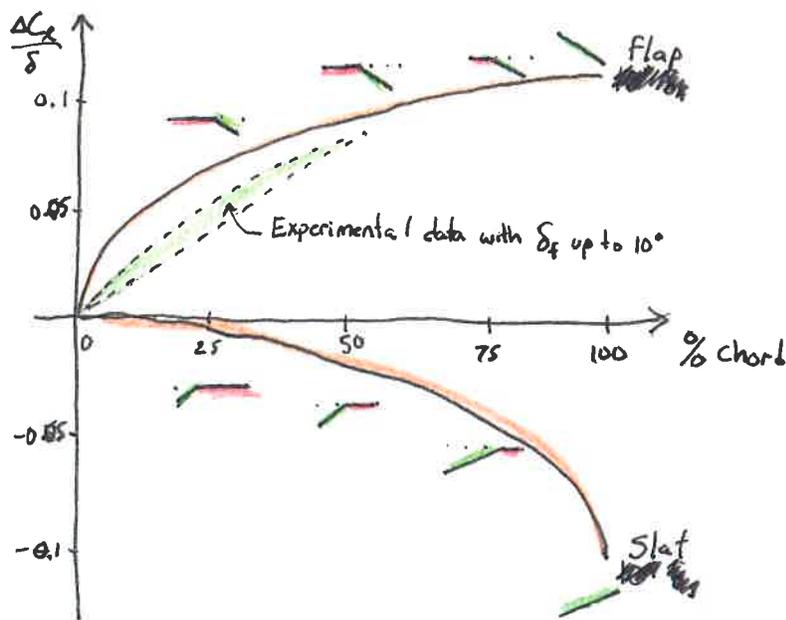
Why use slat? Defer to later lesson after discussion separation. Useful/detrimental!!

For the purely inviscid TAT, how effective are slats and flaps at generating lift?



- For a given configuration (fixed  $\theta_s$  and  $\theta_f$ ) flaps shift the  $C_L$  curve up.
- For a given configuration, slats slightly shift the  $C_L$  curve down.

Effectiveness of slats and flaps versus chord



From a configuration design perspective, small chord flaps are surprisingly effective. A 16% c flap has half the effect as rotating the entire airfoil. Diminishing returns.

Small chord slats decrease the constant AOA lift only slightly.

A 25% c slat only reduces the lift by 6% compared to rotating the entire airfoil by  $\delta_s$ .

Would we want negative slats for high lift? No!

It would appear that flaps are considerably better at ~~high~~ generating high lift than slats.

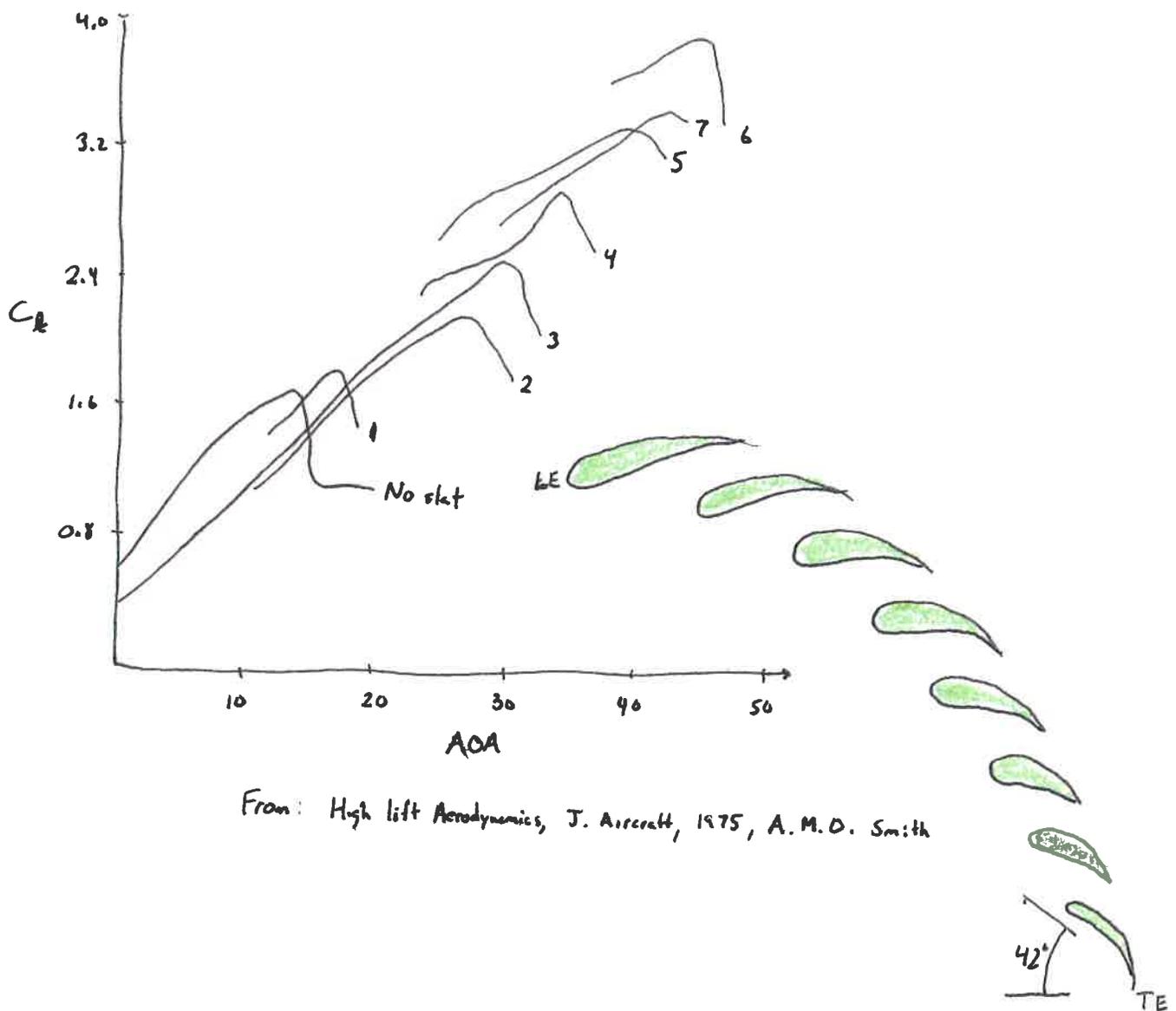
Once we cover viscous flows and separation, we will see why slats are a powerful tool for generating high lift. This occurs not through an upward shift in  $C_L$ , but a right shift in the AOA at stall.

We will discuss high lift strategies and configurations later in this class. Since high lift is so dependent on flow separation and attachment, we will delay further discussion until after we study boundary layers and, in general, viscous airfoil flows.

As a teaser, the following figure is why we use slats

No slat  $C_{Lmax} \approx 1.7$   
at  $13-14^\circ$

8 element  $C_{Lmax} \approx 4.0$   
at  $42^\circ$

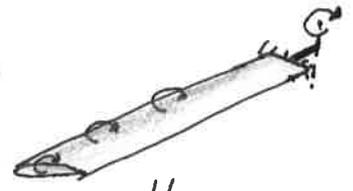


From: High lift Aerodynamics, J. Aircraft, 1975, A.M.D. Smith

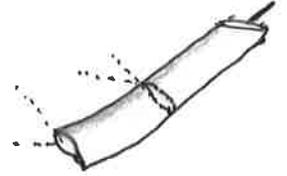
Design a rotorcraft airfoil with  $C_{L_0} = \frac{\pi}{10}$  and

$$C_{m_{x_1}} = 0$$

why?  
Twisting  
blades  
reduce  
performance



Aeroelastic  
twisting...



$$C_{L_0} = \frac{\pi}{10} = \pi A_1 \Rightarrow A_1 = \frac{1}{10}$$

$$C_{m_{x_1}} = 0 = -\frac{\pi}{4}(A_1 - A_2) \Rightarrow A_2 = -\frac{1}{10}$$

$$\text{Implies } A_0 = \alpha = 0 \Rightarrow \int_0^\pi \frac{dz}{dx} d\theta = 0$$

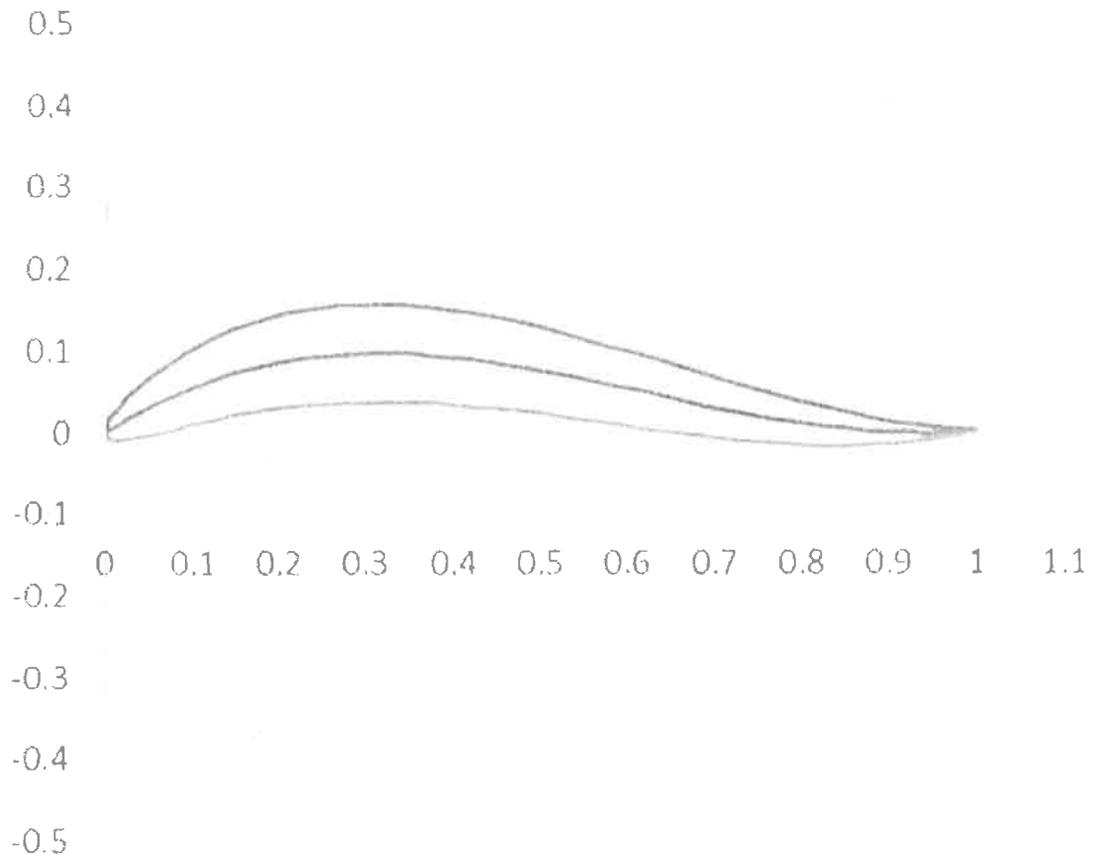
$$Z(0) = 0 \text{ and } Z(\pi) = 0$$

$$A_1 = \frac{1}{10} \Rightarrow \frac{1}{10} = \frac{2}{\pi} \int_0^\pi \frac{dz}{dx} \cos \theta d\theta$$

$$A_2 = -\frac{1}{10} \Rightarrow -\frac{1}{10} = \frac{2}{\pi} \int_0^\pi \frac{dz}{dx} \cos 2\theta d\theta$$

What  $\frac{dz}{dx}$  satisfies these 4 constraints?

4 equations ... 4 terms



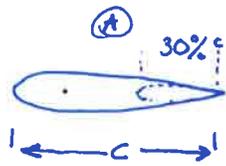
**NACA 6-H-10 Rotorcraft**

Lesson 13 part 5

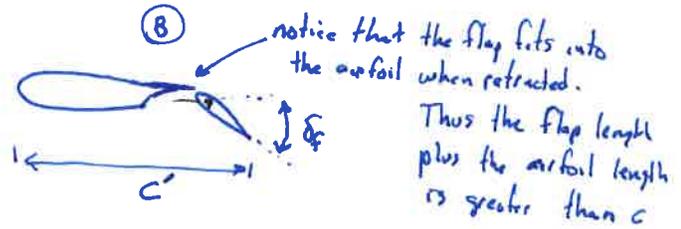
Example: Fowler Flap

# Example: Thin Airfoil Theory

Fowler flaps not only have a trailing edge deflection, but also extend aft.



$\Rightarrow$



Also, this aft section has a camber line angled to the upper surface!



$$\textcircled{A} \quad A_0 = \alpha - \frac{1}{\pi} \int_0^\pi \frac{dz}{dx} d\theta$$

$$= \alpha$$

$$z=0 \Rightarrow \frac{dz}{dx} = 0$$

$$A_n = \frac{2}{\pi} \int_0^\pi \frac{dz}{dx} \cos n\theta d\theta = 0$$

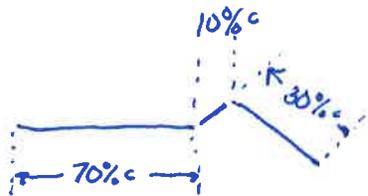
$$C_L = 2\pi A_0 + \pi A_1$$

$$C_L = 2\pi \alpha$$

$$C_m = -\frac{\pi}{4} (A_1 - A_2) = 0 = C_{m,air}$$

Neutral point / Aerodynamic Center at  $1/4$

$\textcircled{B}$



$$z=0$$

$$0.30c \cdot \cos(\delta_f)$$

$$\frac{dz}{dx} = -\tan(\delta_f)$$

$$c' = 0.8 + 0.3 \cos(\delta_f)$$

For the purposes of this analysis, assume:

- 1) The camber line of the aft main airfoil is zero
- 2) The flap deflection is ~~small~~ small, but we will consider the cosine component of length along the  $x$  axis. The sine component is ignored

The transition is at  $80\%c$ .

$$x = \frac{c}{2} (1 - \cos \theta)$$

$$0.8 = \frac{0.8 + 0.3 \cos(\delta_f)}{2} \cdot (1 - \cos \theta)$$

Thus solving gives

$$\theta = 2.05 \text{ rad} \approx 117^\circ$$

$$A_0 = \alpha - \frac{1}{\pi} \int_0^{\pi} \frac{dz}{dx} d\theta = \alpha - \frac{1}{\pi} \int_0^{\pi} 2.05 d\theta - \frac{1}{\pi} \int_{2.05}^{\pi} -\tan(\delta_f) d\theta$$

$$= \alpha + \frac{1}{\pi} \delta_f = \alpha + 0.3475$$

$$A_1 = \frac{2}{\pi} \int_0^{2.05} 0 \cos \theta d\theta + \frac{2}{\pi} \int_{2.05}^{\pi} -\delta_f \cos \theta d\theta = \delta_f 0.565$$

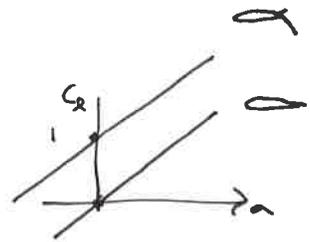
$$A_2 = \frac{2}{\pi} \int_{2.05}^{\pi} -\delta_f \cos 2\theta d\theta = 0.2605 \delta_f$$

$$C_R = 2\pi A_0 + \pi A_1 = 2\pi\alpha + 1.09\delta_f$$

$$C_m = -\frac{\pi}{4}(A_1 - A_2) = -\frac{\pi}{4}(0.565 - 0.2605)\delta_f$$

$$= -0.2392\delta_f$$

$$AC \text{ at } c'/4 = \frac{0.8 + 0.3 \cos(\delta_f)}{4}$$



Fair comparison:

The chord length is different from (A) and (B). For a fair comparison, we will take the (A) length as the standard and compare.

(A)  $C_R = 2\pi\alpha$   $C_m = 0$

(B)  $C'_R = \frac{L}{8c'} = \frac{L}{8c(\frac{c'}{c})} = \frac{L}{8c} \left(\frac{c}{c'}\right) = C_R \left(\frac{c}{c'}\right) \Rightarrow C_R = C'_R \frac{c'}{c}$

$$C_{R_c} = (2\pi\alpha + 1.09\delta_f) \cdot (0.8 + 0.3 \cos \delta_f)$$

$$C_{m_{c'}} = (-0.2392\delta_f) (0.8 + 0.3 \cos \delta_f)^2$$

A.C. at  $\frac{0.8 + 0.3 \cos(\delta_f)}{4}$ , for a shift

$$\frac{0.8 + 0.3 \cos(\delta_f)}{4} - \frac{1}{4} = \frac{-0.2 + 0.3 \cos \delta_f}{4}$$

So adding chord increases the nominal  $C_R$  beyond just the  $C'_R$  term

Similar for  $C_m$  except  $c^2$ !

Aft shift in A.C.

# Aerodynamic Center from Experimental Data

a.c.  $\equiv$  the location where  $C_m$  is constant with  $C_L$

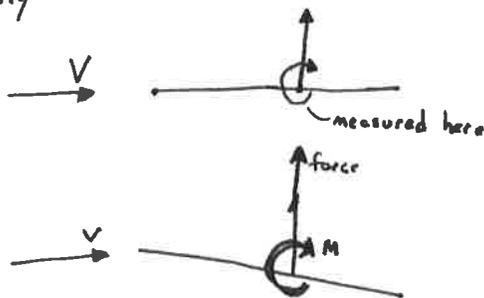
$$\frac{dC_m}{dC_L} = 0$$

For an airplane, we call the a.c. by a different name, the neutral point. The concept is the same.

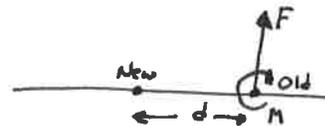
Non-dim:

$$C_L \equiv \frac{L}{\rho c^2} \quad C_m \equiv \frac{M}{\rho c^2} \quad \Rightarrow \quad \frac{dC_m}{dC_L} \frac{\rho c^2}{\rho c^2} = \frac{dM}{dL} = \text{distance}$$

Visually



move measurement point



$$F_d = F$$

$$M_d = M - F \cdot d$$

negative sign

If the moment increases with force, where can we move the measurement point such that the increase in M wrt L decreases?

$$\text{want } \frac{dM}{dF} = 0 \Rightarrow \frac{d}{dF}(M - F \cdot d) = -d$$

The a.c. is at:

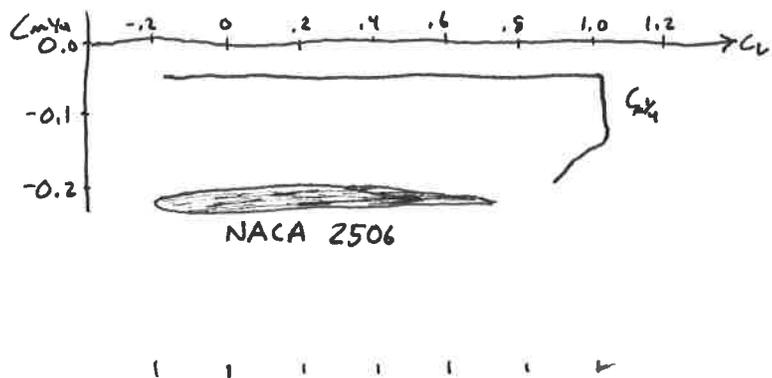
$$a.c. = x_{\text{measured}} - \frac{dC_m}{dC_L} c_{\text{measured}}$$

Usually, the measurement point is at the  $1/4 c$  ("quarter chord") or MAC if an airplane.

$$a.c. = \frac{1}{4} c - \frac{dC_m}{dC_L} c$$

mean aerodynamic chord.

# Experimental Data



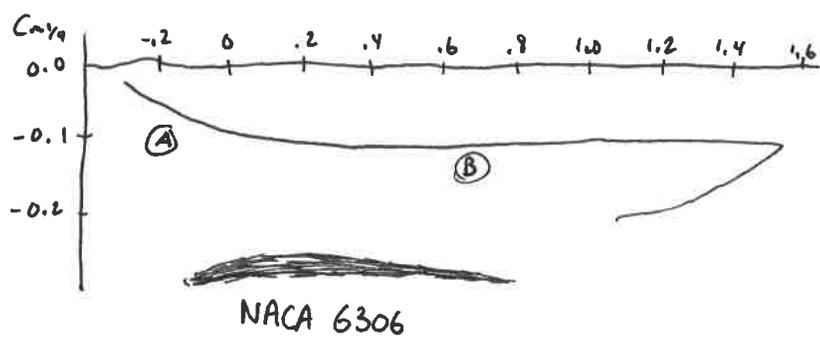
$$C_{m_0} (c_c=0) \approx -0.05$$

$$\frac{dC_m}{dc_c} (c_c=0) \approx 0$$

$$a.c. = \frac{1}{4} - 0 = 0.25$$

At stall,  $\frac{dC_m}{dc_c}$  is strongly negative.  
 the a.c. at stall shifts aft.  
 After stall,  $\frac{dC_m}{dc_c}$  is strongly positive!!  
 and  $C_m$  is negative.

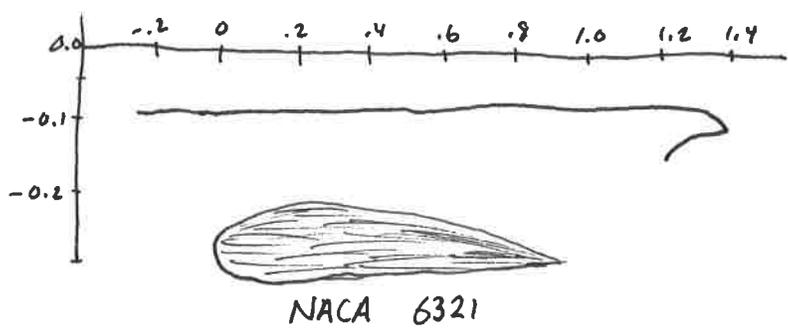
No warning and nasty behavior... avoid !!?



At (A),  $\frac{dC_m}{dc_c} \approx \frac{\Delta C_m}{\Delta c_c} = \frac{-0.1 - 0}{0 - -0.4} = \frac{-0.1}{0.4}$

$$a.c. = \frac{1}{4}c + \frac{1}{4}c = \frac{1}{2}c$$

At (B)  $\frac{dC_m}{dc_c} \approx 0$      a.c. =  $\frac{1}{4}c$



$$\frac{dC_m}{dc_c} \approx \frac{\Delta C_m}{\Delta c_c} = \frac{0.025}{1.4} \approx 0.018$$

$$a.c. = \frac{1}{4}c - 0.018c$$

$$a.c. = 23\%c$$

## Joukowski:

$$a.c. \approx \frac{1}{4} + \frac{t^2}{2}$$

$$\approx \frac{1}{4} + \frac{1}{4} \left(\frac{t}{c}\right)^2$$

$$a.c._{21\% \text{ thick}} \approx 26\%$$

a.c. shift with  $\frac{t}{c}$  for the  
 NACA 4 and 5 digit airfoils  
 is the opposite of Joukowski!