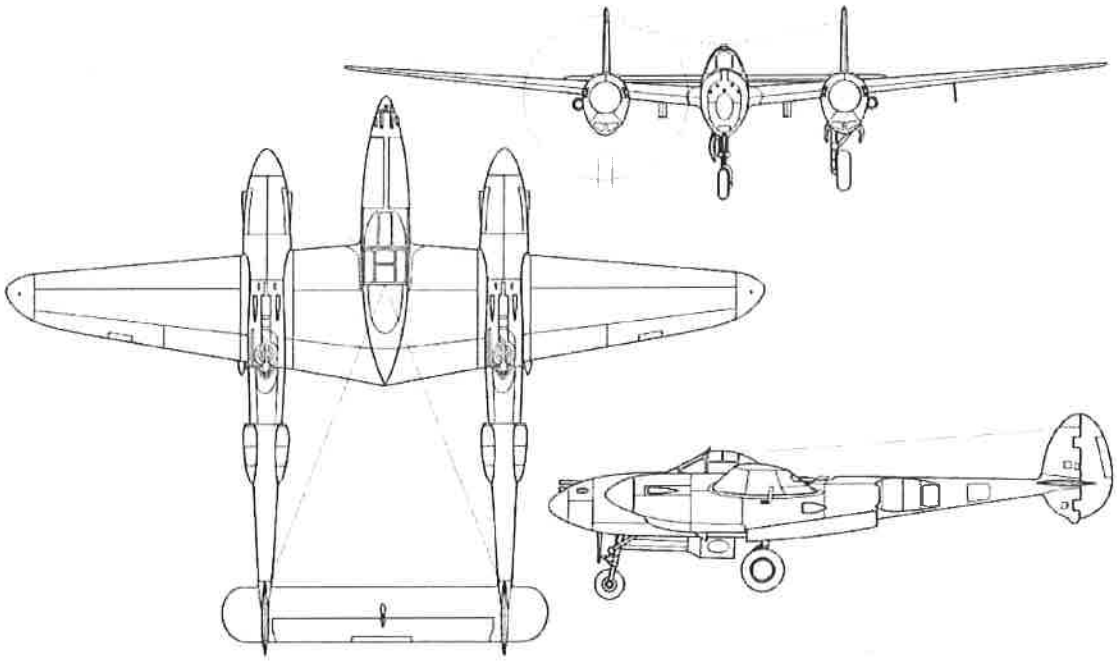
Lesson 24

Subsonic Compressible Flow

Prandtl Glauert

Lockheed P-38:
Twin engine “heavy” fighter from pre-WW2
414 mph at FL250



5 November 1941
Compressibility Kills

WEDNESDAY MORNING. Los Angeles Times • NOVEMBER 5, 1941. B

Pursuit Plane Crashes Into Glendale House After Breakup in Air

P-38 Test Plane Falls and Burns

Pilot of Military Craft Killed by Dive Into Glendale Home

Continued from First Page

bedroom of the home at the time of the crash. He attempted to help rescue Virden. Glendale city firemen were dousing the wreckage with water an hour after the crash.

Virden resided at 4541 Ben-

WARTIME REPORT

ORIGINALLY ISSUED

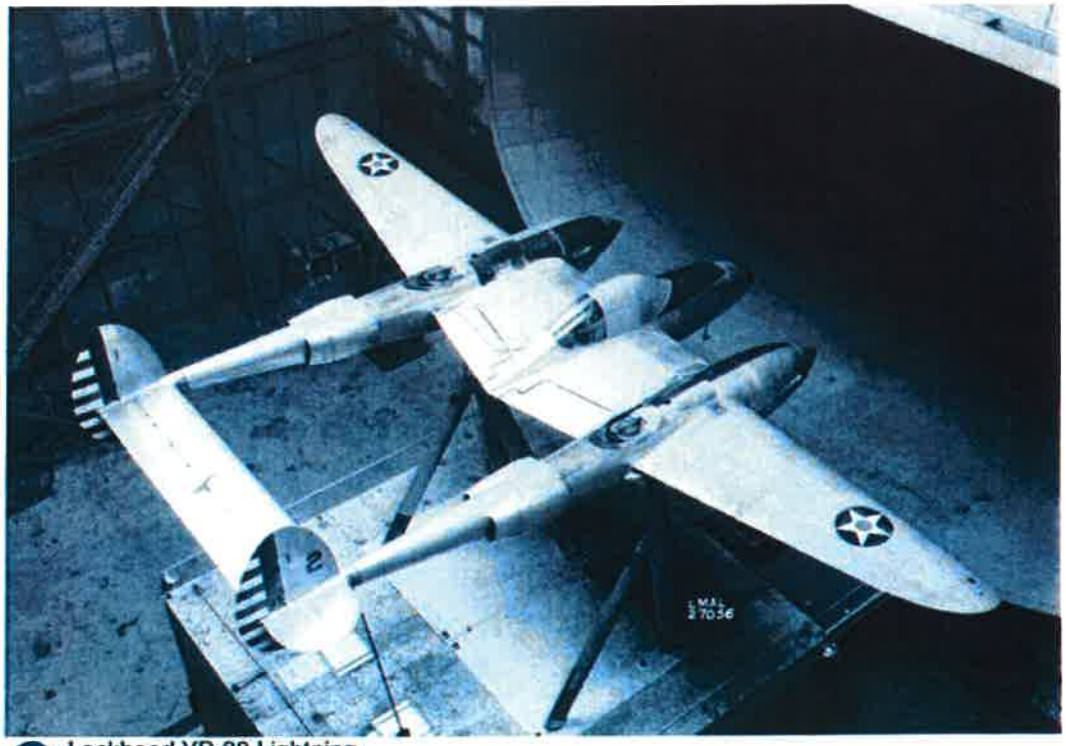
October 1942 as
Memorandum Report

INVESTIGATION OF DIVING MOMENTS OF A PURSUIT AIRPLANE

IN THE AMES 16-FOOT HIGH-SPEED WIND TUNNEL

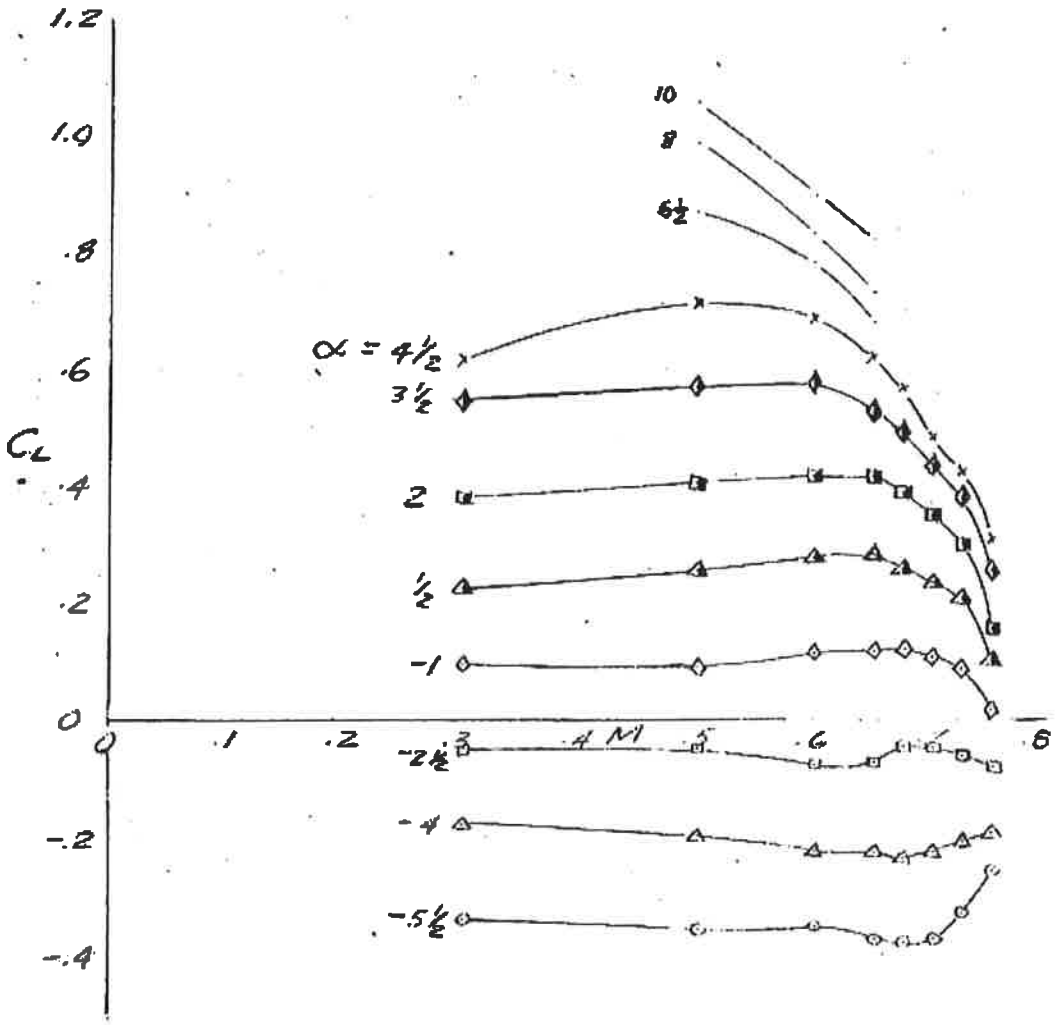
By Albert L. Erickson

- Mach Tuck. Flutter. Tail breaks off.
- How do you fight a war when your aircraft are killing pilots?
- Pilots were afraid to engage adversary aircraft for fear of an uncontrollable dive.



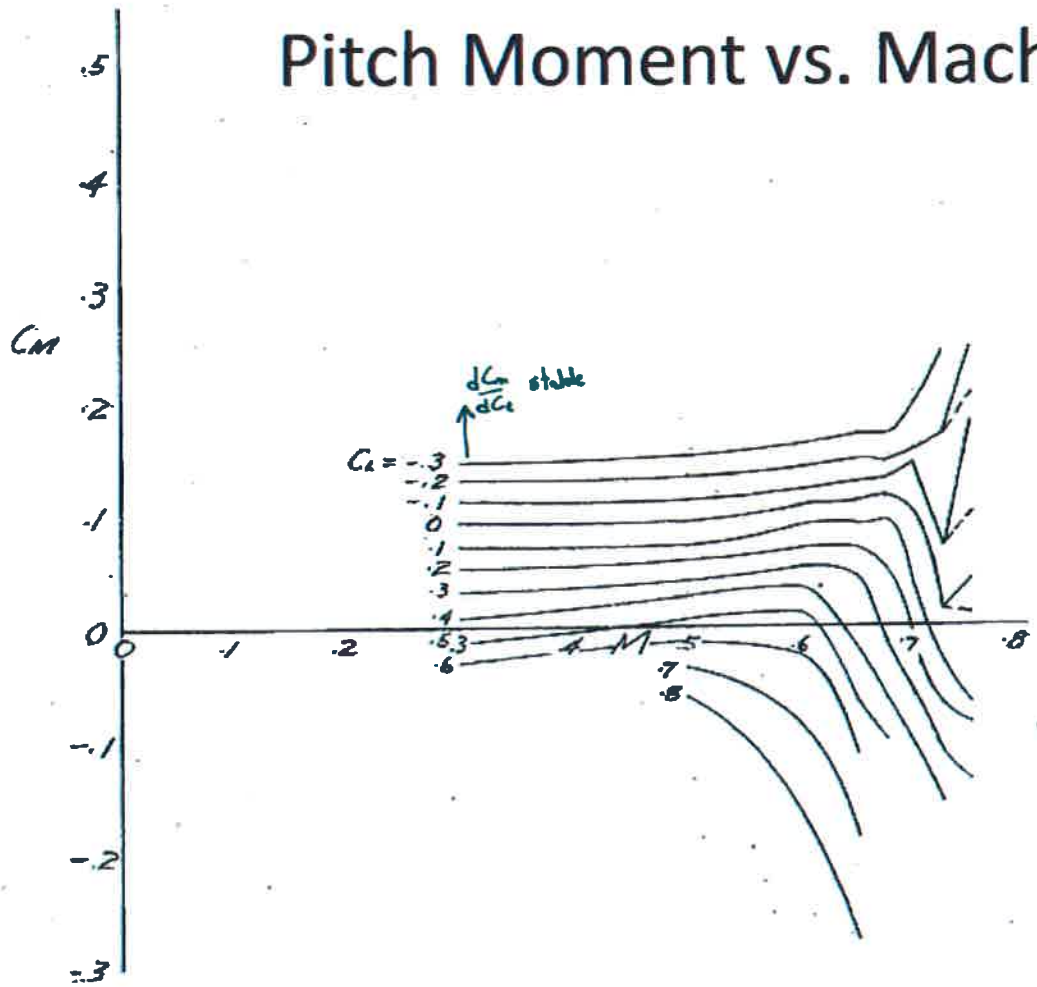
Lockheed YP-38 Lightning
NASA Langley Research Center 12/31/1942 Image # EL-2001-00380

Lift vs. Mach



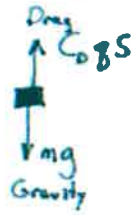
NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

Pitch Moment vs. Mach



NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

Dive Velocity



$$mg = C_D q S \Rightarrow q = \frac{mg}{C_D S} = \frac{W}{C_D S} = \frac{1}{C_D}\underbrace{\frac{W}{S}}_{\text{wing loading}}$$

$$C_D = C_D(M)$$

$$q = q(M, h) = \frac{1}{2}\rho V^2 = \frac{1}{2}\gamma p M^2$$

P38: $W \approx 18000\ \text{lbf}$, $S \approx 327\ \text{ft}^2$ $\Rightarrow \frac{W}{S} \approx 55\ \text{psf} \approx 0.38\ \text{psi}$

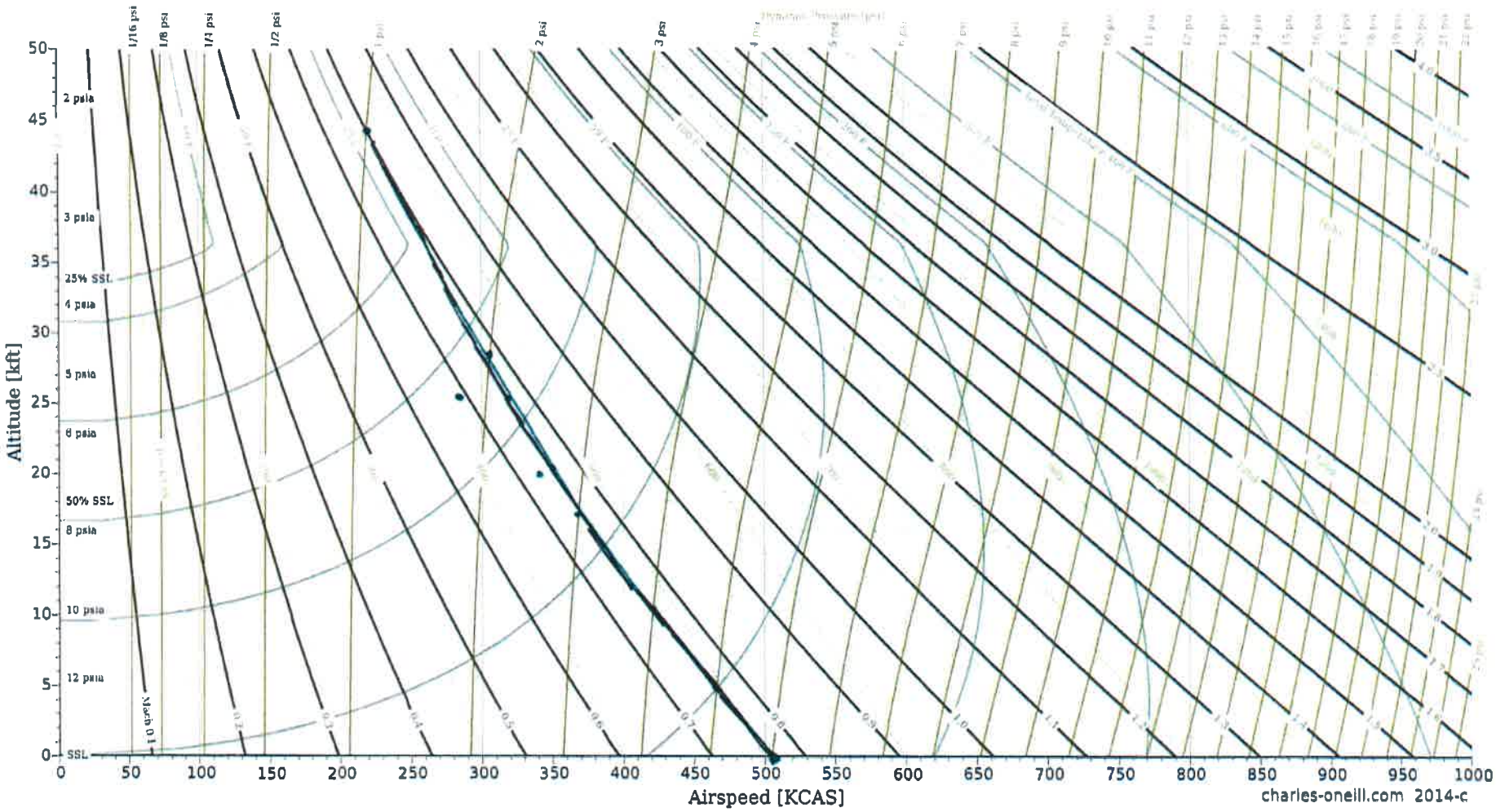
For equilibrium

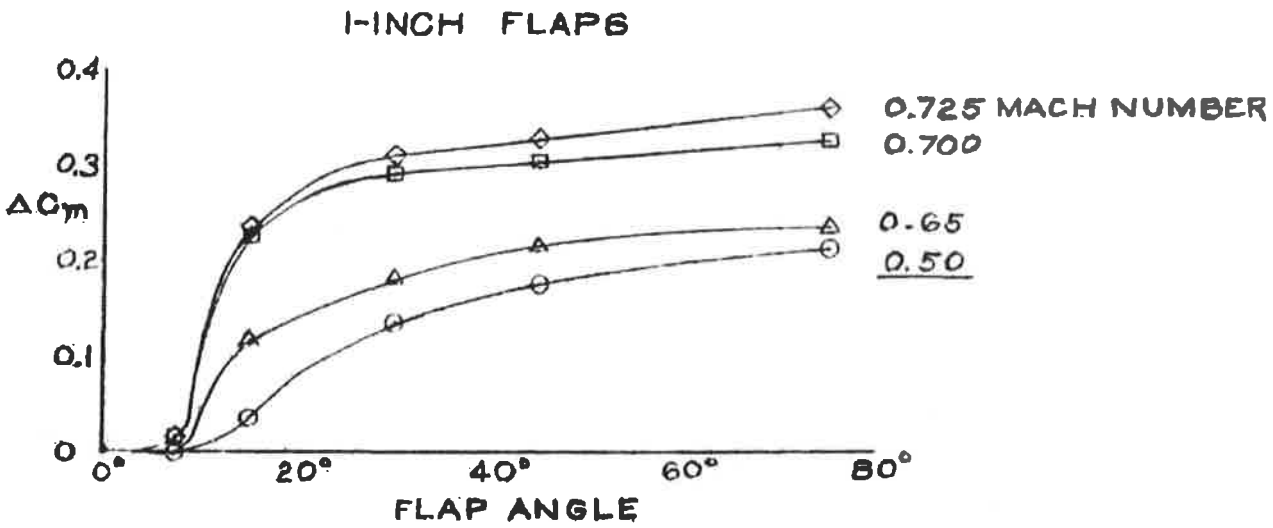
$$C_D = 0.381^{\text{psi}} \cdot \frac{1}{q^{[\text{psi}]}}$$

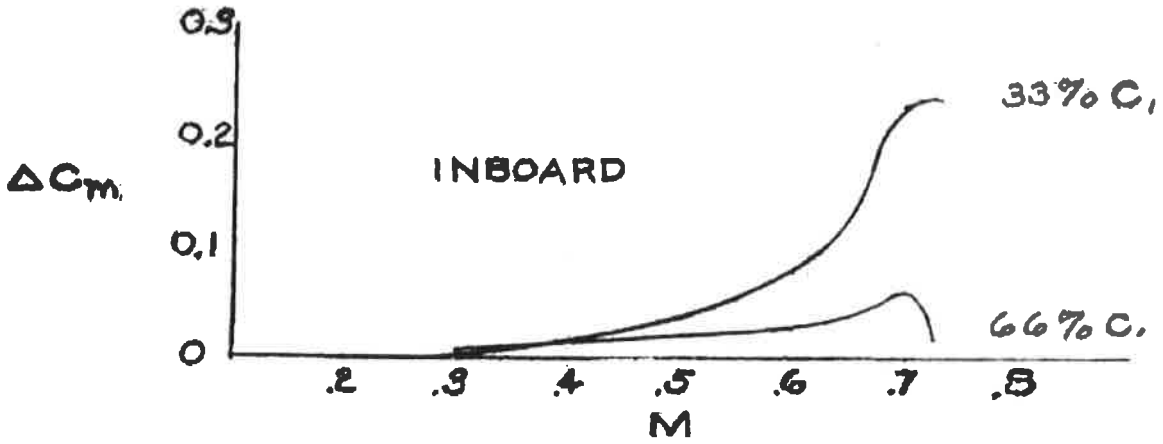
$$q = 2\ \text{psi} \Rightarrow C_D = 0.19 \Rightarrow M \approx 0.78$$

$\alpha = 0$

q^{psi}	C_D	M	h[ft]
1	0.3800	0.8	44000
2	0.1900	0.78	27000
2.5	0.1500	0.75	
3.0	0.1270	0.75	17000
4.0	0.0950	0.74	
5.0	0.0760	0.74	4000
6.0	0.0640	0.73	← Sea level


Altitude [kft]
Airspeed [KCAS]
1/16 psi
1/8 psi
1/4 psi
1/2 psi
2 psi
3 psi
2 psia
3 psia
25% SSL
4 psia
5 psia
6 psia
50% SSL
8 psia
10 psia
12 psia
SSL
Mach 0.1
charles-oneill.com 2014-c


1-INCH FLAPS
0.4
0.3
0.2
0.1
0
ΔCm
0°
20°
40°
60°
80°
FLAP ANGLE
0.725 MACH NUMBER
0.700
0.65
0.50


0.3
0.2
0.1
0
ΔCm
INBOARD
33% C,
66% C,
.2
.3
.4
.5
.6
.7
.8
M

WARTIME REPORT

ORIGINALLY ISSUED

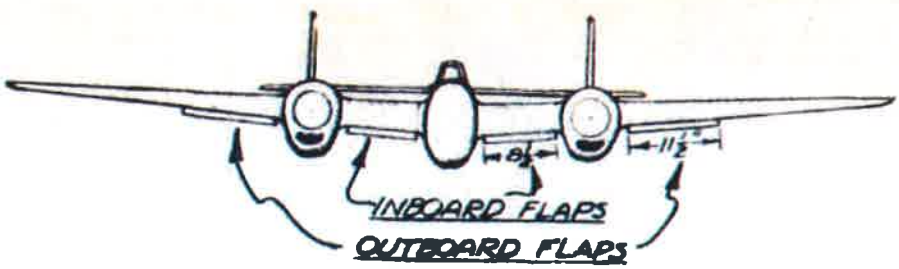
April 1943 as
Memorandum Report 3F12

WIND-TUNNEL INVESTIGATION OF DEVICES FOR IMPROVING THE DIVING CHARACTERISTICS OF AIRPLANES

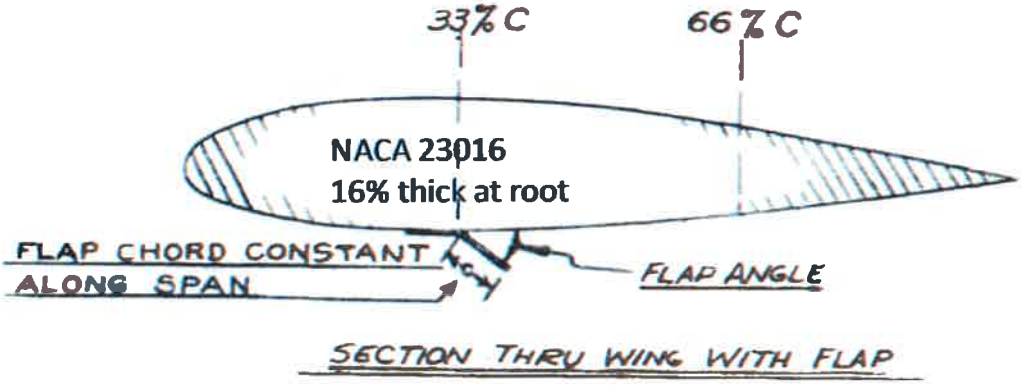
By Albert L. Erickson

Ames Aeronautical Laboratory
Moffett Field, California

FILE COPY
To be returned to
the files of the National
Advisory Committee



NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS



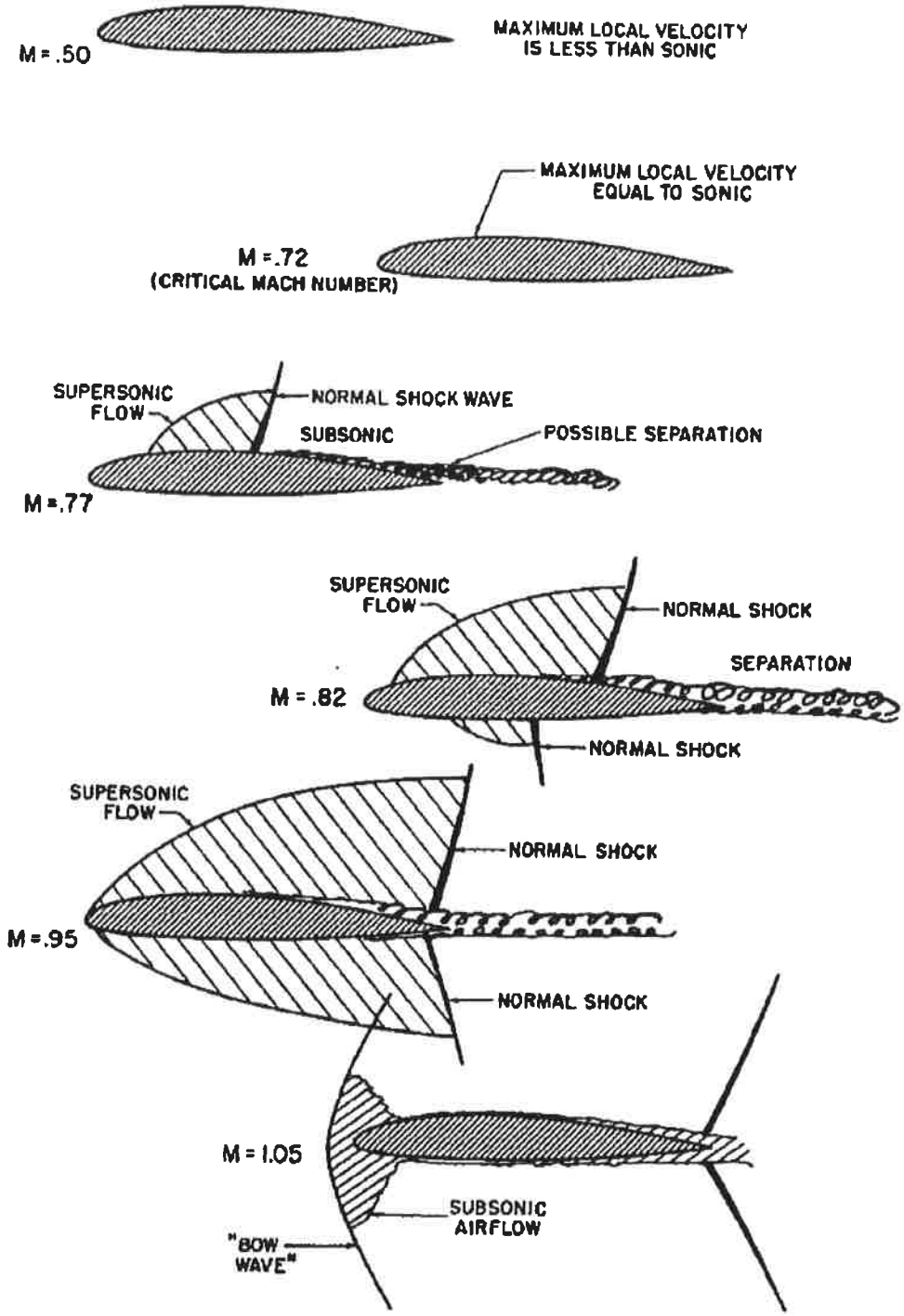
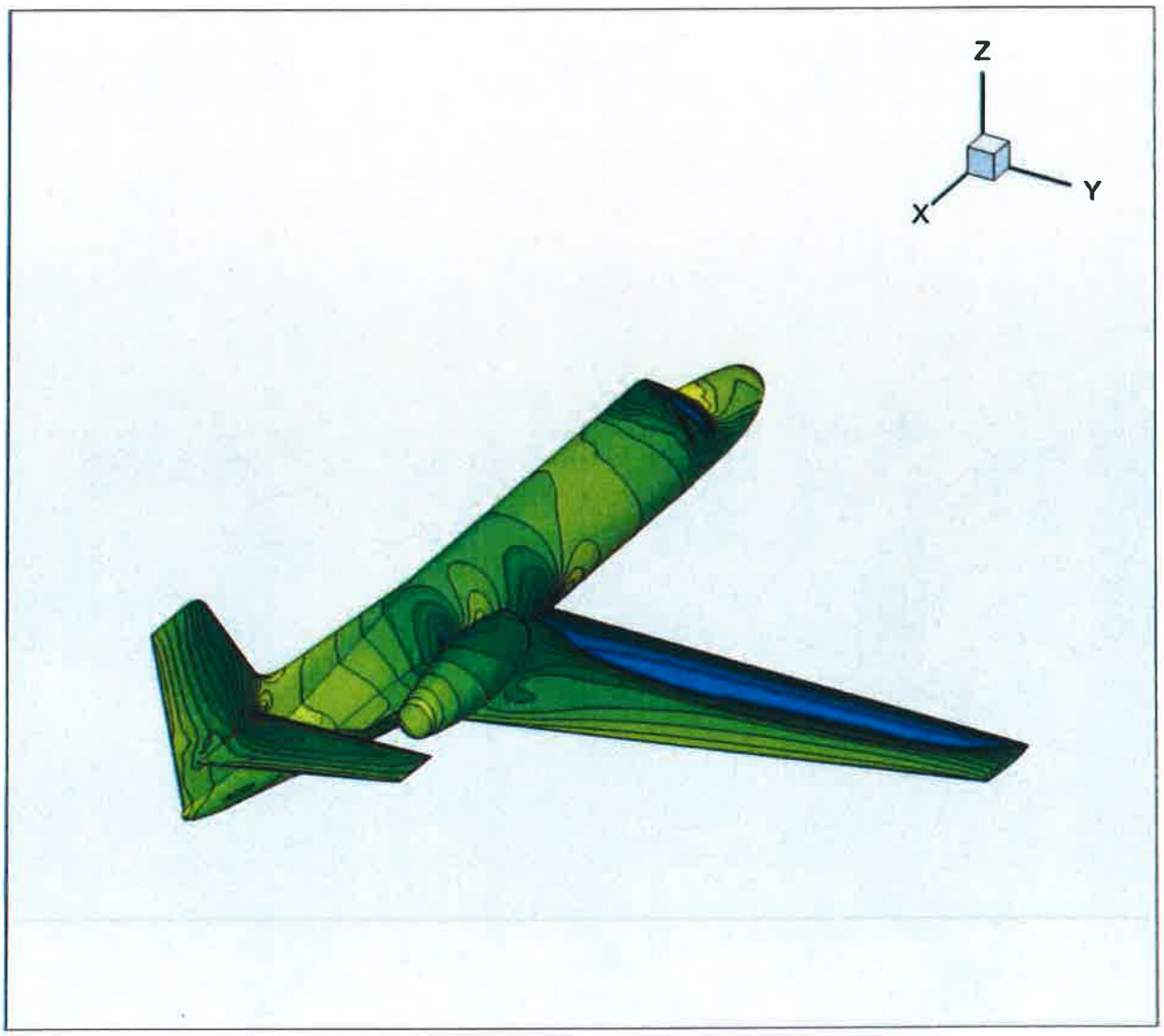
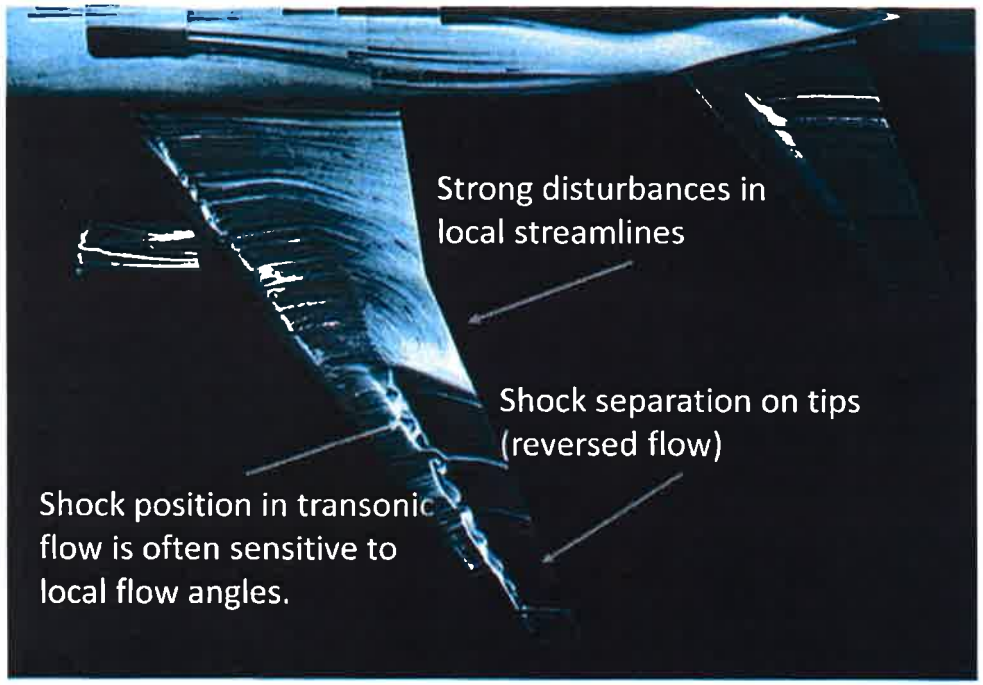


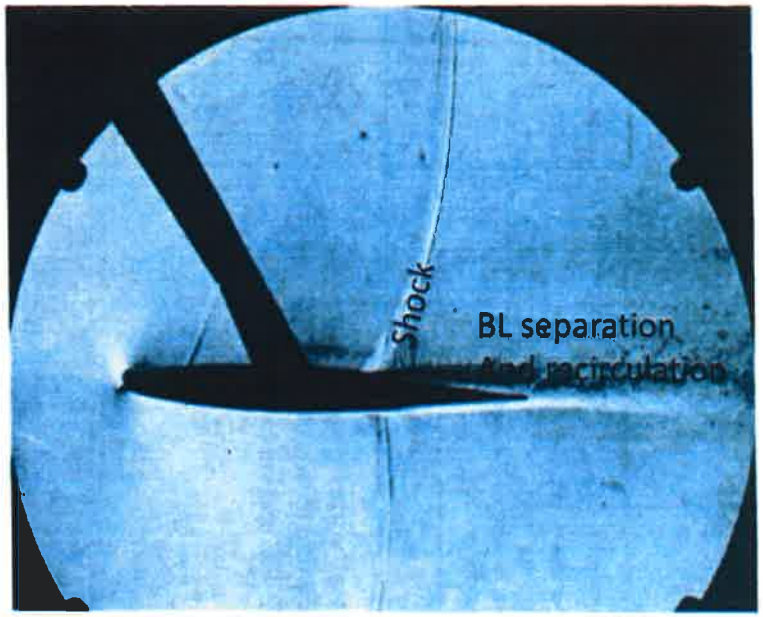
Figure 3.9. Transonic Flow Patterns (sheet 1 of 2)


Z
Y
X

Physics of Phenomenon:

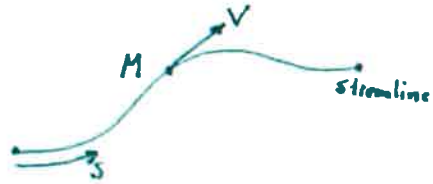


M= 0·83 (UNCORRECTED)



A compressible flow contains large variations in density. Previously, the mass continuity equation was used to show that the source density for all flows scale as:

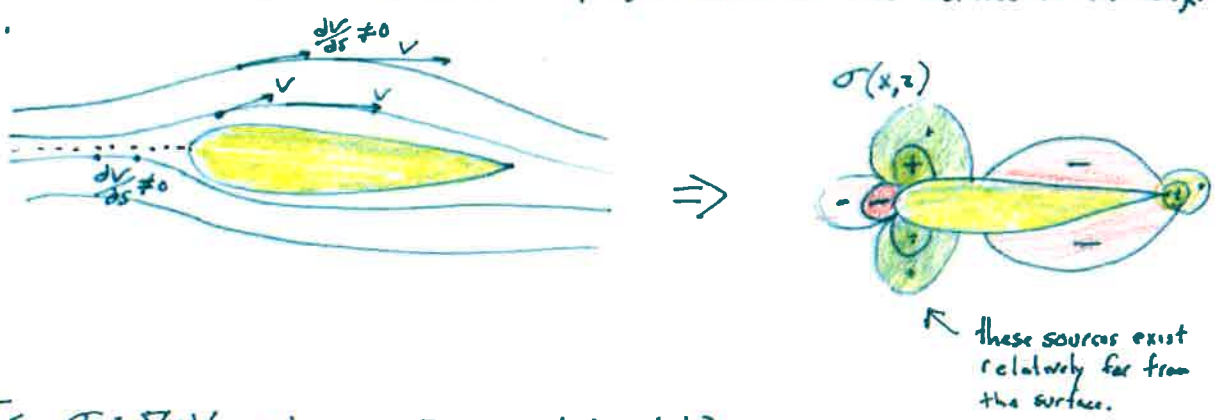
$$\nabla \cdot \vec{V} = \sigma = M^2 \frac{\partial V}{\partial s}$$



So all flows are compressible, but when M^2 is near zero, the flow can be considered incompressible.

Notice that $\sigma = M^2 \frac{\partial V}{\partial s}$ occurs throughout the flow (not just at the surface).

The thin airfoil theory assumption of lumping sources to the surface is no longer valid.



Q: Is $\sigma = \nabla \cdot V$ and $\omega = \nabla \times V$ still valid?

Q: Is the reverse operator still valid?

$$V = V_b + V_\sigma + V_\omega$$

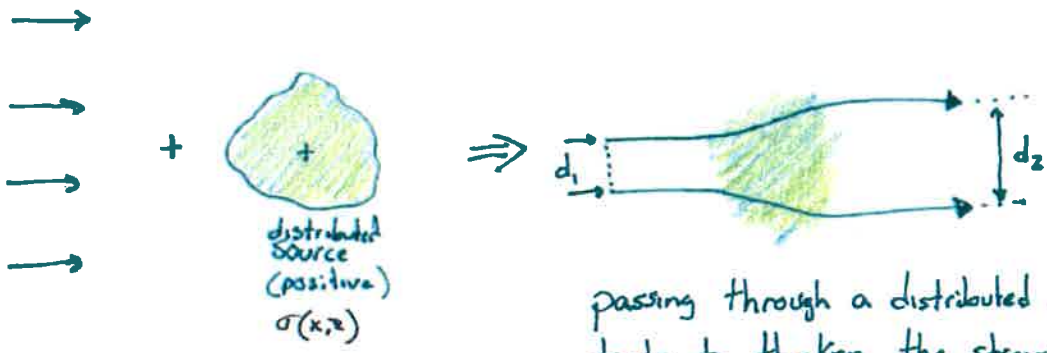
$$= V_b + \frac{1}{4\pi}\iiint \sigma(r') \frac{r-r'}{|r-r'|^3} dx'dy'dz' + \frac{1}{4\pi}\iiint \omega(r') \times \frac{r-r'}{|r-r'|^3} dx'dy'dz'$$

A: Yes and Yes

Review of sources and sinks in flows:

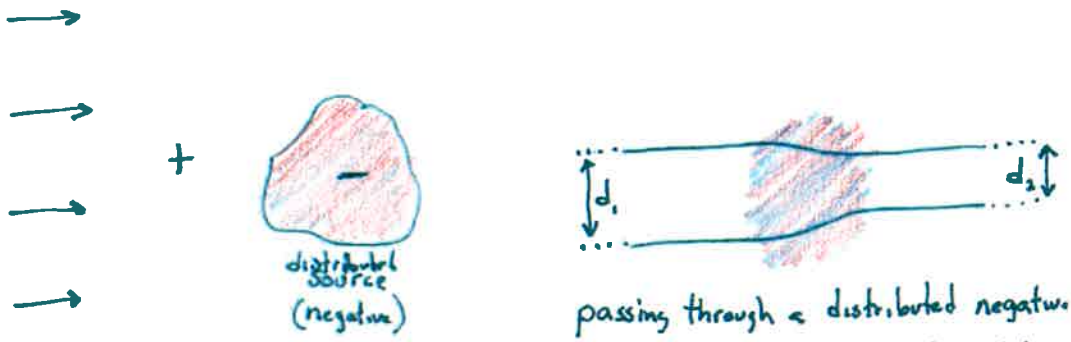
1)

2)



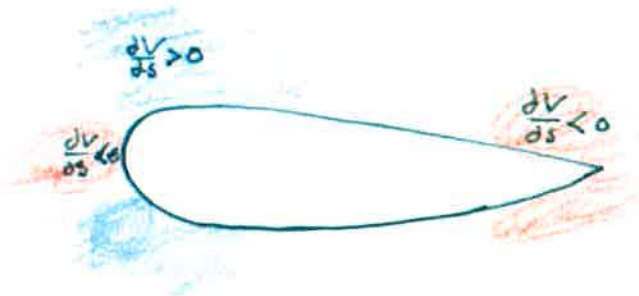
passing through a distributed source (positive) tends to <u>thicken</u> the streamtube

3)



passing through a distributed negative source tends to <u>thin</u> the streamtube

For a subsonic airfoil, where are the acceleration and deceleration regions?

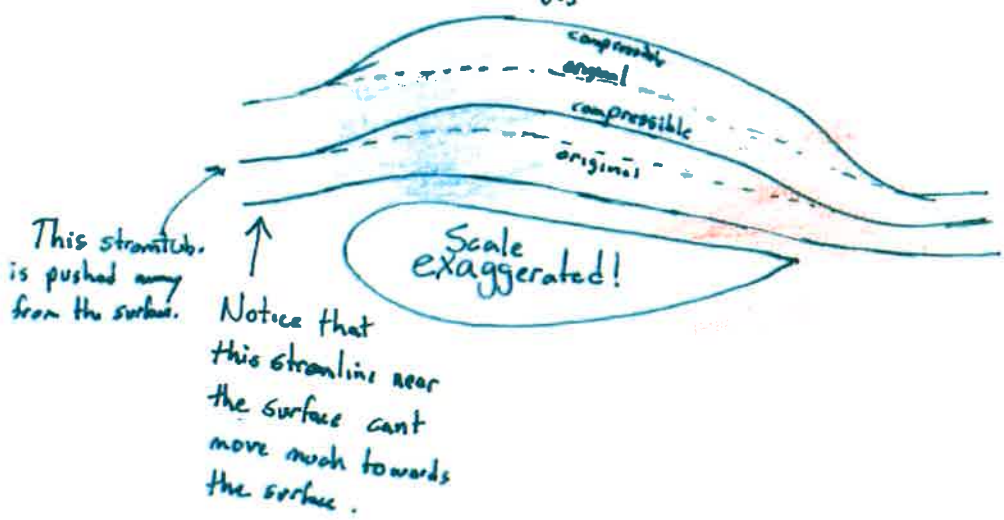


$$\sigma = M^2 \frac{dV}{ds} \Rightarrow \sigma > 0 \text{ when } \frac{dV}{ds} > 0$$
$$\sigma < 0 \text{ when } \frac{dV}{ds} < 0$$

Thus

Streamtubes thicken when $\frac{dV}{ds} > 0$

Streamtubes thin when $\frac{dV}{ds} < 0$



Also, the positive source tends to slow the flow upstream and speed the flow downstream.

In comparison to subsonic incompressible flows, the streamtubes thicken and the velocity increases in a compressible flow.

$$\boxed{\text{Compressible flows tend to magnify both perturbations and the distance at which the disturbances act.}}$$

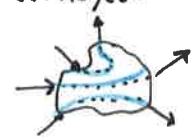
Velocity Potential Equation

$$V = \nabla \phi = \left(\underbrace{\frac{d\phi}{dx}}_{u}, \underbrace{\frac{d\phi}{dy}}_{v}, \underbrace{\frac{d\phi}{dz}}_{w} \right)$$

- The continuity equation is

$$\nabla \cdot (\rho \vec{V}) = 0$$

this is just another way of saying that mass is neither created nor destroyed. (what goes in must go out)



Expand this to

$$\frac{d}{dx}(\rho u) + \frac{d}{dy}(\rho v) + \frac{d}{dz}(\rho w) = 0$$

Expand with the chain rule

$$\rho \frac{du}{dx} + \frac{d\rho}{dx} u + \rho \frac{dv}{dy} + \frac{d\rho}{dy} v + \rho \frac{dw}{dz} + \frac{d\rho}{dz} w = 0$$

Substitute $u = \frac{d\phi}{dx}$ and $v = \frac{d\phi}{dy}$ and $w = \frac{d\phi}{dz}$

$$\rho \frac{d^2\phi}{dx^2} + \frac{d\rho}{dx}\frac{d\phi}{dx} + \rho \frac{d^2\phi}{dy^2} + \frac{d\rho}{dy}\frac{d\phi}{dy} + \rho \frac{d^2 w}{dz^2} + \frac{d\rho}{dz}\frac{d\phi}{dz} = 0$$

Collect terms

$$\rho \left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} \right) + \frac{d\rho}{dx}\frac{d\phi}{dx} + \frac{d\rho}{dy}\frac{d\phi}{dy} + \frac{d\rho}{dz}\frac{d\phi}{dz} = 0$$

- From the same place that Bernoulli's equation comes from, the inviscid form of the momentum equation is:

$$-\frac{dp}{\rho} = \frac{1}{2} d(V^2) = \frac{1}{2} \cdot 2 \cdot V\,dV = V\,dV$$

or

$$dp = \frac{1}{2} \rho\, d(u^2 + v^2) = \frac{1}{2} \rho\, d\left(\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2 \right)$$

In AEM 413, you will study compressible flow. There, you will find that

$$\frac{dp}{d\rho} = \underbrace{a^2}_{\text{speed of sound squared}}$$

in air at SSL, $a \approx 1100 \ ft/s$

$a^2 \approx 1.2 \times 10^6 \ ft^2/s^2$

Thus, small changes in density <u>create</u> <u>large</u> changes in pressure! This is only valid for isentropic flows ... no shocks...

Substitute into momentum eqn.

$$dp = -\frac{1}{2}\rho \, d\underbrace{\left(\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2\right)}_{\updownarrow} = -d\rho \, a^2$$

or

$$dp = -\frac{1}{2}\frac{\rho}{a^2} d(\cdots)$$

From the mass eqn, we need $\frac{d\rho}{dx}, \frac{d\rho}{dy}, \frac{d\rho}{dz}$

$$\frac{d\rho}{dx} = -\frac{1}{2}\frac{\rho}{a^2}\frac{d}{dx}(\cdots) = -\frac{1}{2}\frac{\rho}{a^2}\frac{d}{dx}\left(\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2\right)$$

Chain rule

$$\frac{d\rho}{dx} = -\frac{1}{2}\frac{\rho}{a^2}\left[2\frac{d\phi}{dx}\frac{d}{dx}\left(\frac{d\phi}{dx}\right) + 2\frac{d\phi}{dy}\frac{d}{dx}\left(\frac{d\phi}{dy}\right) + 2\frac{d\phi}{dz}\frac{d}{dx}\left(\frac{d\phi}{dz}\right)\right]$$

$$= -\frac{\rho}{a^2}\left[\frac{d\phi}{dx}\frac{d^2\phi}{dx^2} + \frac{d\phi}{dy}\frac{d^2\phi}{dx\,dy} + \frac{d\phi}{dz}\frac{d^2\phi}{dx\,dz}\right]$$

Likewise

$$\frac{d\rho}{dy} = -\frac{\rho}{a^2}\left[\frac{d\phi}{dx}\frac{d^2\phi}{dx\,dy} + \frac{d\phi}{dy}\frac{d^2\phi}{dy^2} + \frac{d\phi}{dz}\frac{d^2\phi}{dy\,dz}\right]$$

$$\frac{d\rho}{dz} = -\frac{\rho}{a^2}\left[\frac{d\phi}{dx}\frac{d^2\phi}{dx\,dz} + \frac{d\phi}{dy}\frac{d^2\phi}{dy\,dz} + \frac{d\phi}{dz}\frac{d^2\phi}{dz^2}\right]$$

Notice that these are non-linear terms (as the mom' eqn is...)

Substitute, mass and momentum terms.

$$\rho\left(\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}\right)+\frac{d\rho}{dx}\frac{d\phi}{dx}+\frac{d\rho}{dy}\frac{d\phi}{dy}+\frac{d\rho}{dz}\frac{d\phi}{dz}=0$$

$$\frac{\rho}{a^2}\left(\boxed{\frac{d\phi}{dx}\frac{d^2\phi}{dx^2}}+\frac{d\phi}{dy}\frac{d^2\phi}{dxdy}+\frac{d\phi}{dz}\frac{d^2\phi}{dxdz}\right)$$

$$\frac{\rho}{a^2}\left(\frac{d\phi}{dx}\frac{d^2\phi}{dxdy}+\boxed{\frac{d\phi}{dy}\frac{d^2\phi}{dy^2}}+\frac{d\phi}{dz}\frac{d^2\phi}{dydz}\right)$$

$$\frac{\rho}{a^2}\left(\frac{d\phi}{dx}\frac{d^2\phi}{dxdz}+\frac{d\phi}{dy}\frac{d^2\phi}{dydz}+\boxed{\frac{d\phi}{dz}\frac{d^2\phi}{dz^2}}\right)$$

Density term drops out!

notice that this term could be matched with $\frac{d^2\phi}{dz^2}$ term.

$$\frac{d^2\phi}{dx^2}\left(1-\frac{1}{a^2}\left(\frac{d\phi}{dx}\right)^2\right) \qquad -\frac{2}{a^2}\frac{d\phi}{dy}\frac{d\phi}{dx}\frac{d^2\phi}{dxdy}$$

$$+\frac{d^2\phi}{dy^2}\left(1-\frac{1}{a^2}\left(\frac{d\phi}{dy}\right)^2\right) \qquad +-\frac{2}{a^2}\frac{d\phi}{dx}\frac{d\phi}{dz}\frac{d^2\phi}{dxdz} \qquad =0$$

$$+\frac{d^2\phi}{dz^2}\left(1-\frac{1}{a^2}\left(\frac{d\phi}{dz}\right)^2\right) \qquad -\frac{2}{a^2}\frac{d\phi}{dy}\frac{d\phi}{dz}\frac{d^2\phi}{dydz}$$

Q: If $\frac{d\phi}{dx}=U$, then $\frac{1}{a^2}\left(\frac{d\phi}{dx}\right)^2=\frac{U^2}{a^2}=?$ $\qquad$ Mach $\#_{\text{Undrestin}} \equiv \frac{U}{a}$

Q: For subsonic aircraft (the focus of this class), $M<1$. What happens when $\underbrace{M<<1}_{\text{M much smaller than 1.}}$?

$$\frac{d^2\phi}{dx^2}\left(1-\cancel{M_x^2}^{\,0}\right)+\frac{d^2\phi}{dy^2}\left(1-\cancel{M_y^2}^{\,0}\right)+\frac{d^2\phi}{dz^2}\left(1-\cancel{M_z^2}\right)$$

$$-\cancel{\frac{2}{a^2}(v)(U)}^{\,0}\frac{d^2\phi}{dxdy}-\cancel{\frac{2}{a^2}uw}^{\,\sim 0}\frac{d^2\phi}{dxdz}-\cancel{\frac{2}{a^2}vw}^{\,\sim 0}\frac{d^2\phi}{dydz}=0$$

$$\boxed{\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}=0}$$

Where did we see this before?

potential flows

Linearized Potential Equation

If we say $u = U_\infty + \hat{u}$ and $v = \hat{v}$ and reduce with

$0 < M < 0.8$

or

$M > 1.2$

and

$M \lesssim 5$

An amazing result comes from the Velocity potential equation.

$$\left(1 - M_\infty^2\right)\frac{d^2\hat{\phi}}{dx^2} + \frac{d^2\hat{\phi}}{dy^2} + \frac{d^2\hat{\phi}}{dz^2} = 0$$

Notice that we could have predicted this equation from the velocity potential equation

$$\frac{d^2\phi}{dx^2}\left(1 - \frac{1}{a^2}\left(\frac{d\phi}{dx}\right)^2\right) - \frac{2}{a^2}uv\frac{d^2\phi}{dx\,dy}$$

u^2 large, keep; uv small, drop

$$+ \frac{d^2\phi}{dy^2}\left(1 - \frac{1}{a^2}\left(\frac{d\phi}{dy}\right)^2\right) - \frac{2}{a^2}uw\frac{d^2\phi}{dx\,dz}$$

v^2 small, drop; uw small, drop

$$+ \quad \cdots \qquad - \quad \cdots$$

Keep terms

$$\frac{d^2\phi}{dx^2}\Big(1 - \underbrace{\frac{1}{a^2}u^2}_{\frac{u^2}{a^2} \approx M_\infty^2}\Big) + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0$$

$$\boxed{\frac{d^2\phi}{dx^2}\left(1 - M_\infty^2\right) + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0}$$

The derivation in Anderson is excessively long.....

What does this mean?

- Linear PDE!
- X-direction is scaled by $(1 - M_\infty^2)$
- The type of solution is different at $M < 1$ than $M > 1$!!

M	$(1-M_\infty^2)$
0	1
0.1	0.99
0.3	0.9
0.7	0.5
0.9	0.19
1.0	0

Prandtl - Glauert

Define $\beta^2 = 1 - M_\infty^2$, so $\beta^2 \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0$

Transform the y and z coordinates

$$\eta = \beta y \qquad \tau = \beta z$$

Chain rule

$$\frac{d\phi}{dy} = \frac{d\phi}{d\eta}\frac{d\eta}{dy} = \frac{d\phi}{d\eta}\beta \qquad \text{and} \qquad \frac{d}{dy}\left(\frac{d\phi}{dy}\right) = \frac{d}{d\eta}\left(\frac{d\phi}{d\eta}\frac{d\eta}{dy}\right)\frac{d\eta}{dy}$$

likewise for $\frac{d\phi}{dz}$

$$= \frac{d^2\phi}{d\eta^2}\left(\frac{d\eta}{dy}\right)^2 = \frac{d^2\phi}{d\eta^2}\beta^2$$

Substitute

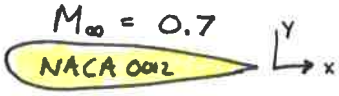
$$\beta^2 \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{d\eta^2}\beta^2 + \frac{d^2\phi}{d\tau^2}\beta^2 = 0$$

if $\beta \neq 0$, then $\boxed{\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{d\eta^2} + \frac{d^2\phi}{d\tau^2} = 0}$

Laplace eqn!
We used this to solve for potential flows at low speeds!

Q: What geometry and governing equation fit the following airfoil?

$M_\infty = 0.7$



$$\underbrace{(1-M_\infty^2)}_{\beta^2 = 0.5}\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0$$

Chord = 1

$t/c = 12\%$

$\Rightarrow$

$M_\infty = 0$!!

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{d\eta^2} = 0$$

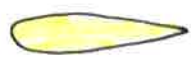
Chord = 1

$t/c = \frac{0.5}{1}(0.12) = 6\%$

We have the capability to solve this!!

C_{p_0}

$\Rightarrow$



transform back

$$C_p = \frac{C_{p_0}}{\beta}$$

Solve this $\Rightarrow$ Apply to this

Prandtl. Glauert (1928)

$$C_p = \frac{C_{p_{M=0}}}{\sqrt{1-M_\infty^2}}$$

2D: $$C_\ell = \frac{C_{\ell_{M=0}}}{\sqrt{1-M_\infty^2}} \qquad C_m = \frac{C_{m_{M=0}}}{\sqrt{1-M_\infty^2}}$$

$$C_{\ell_\alpha} = \frac{C_{\ell_{\alpha\, M=0}}}{\sqrt{1-M_\infty^2}}$$

Q: If C_{ℓ_α} is 2π at $M=0$, what is C_{ℓ_α} at $M=0.5$?

$$C_{\ell_\alpha} = \frac{2\pi}{\sqrt{1-0.5^2}} = \frac{2\pi}{0.866} \approx 1.15 \cdot 2\pi \qquad 15\% \text{ higher}$$

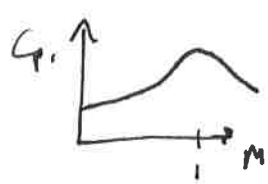
Q: If you were a non-technical writer in 1940 and you saw $C_p = \frac{C_{p_{M=0}}}{\sqrt{1-M_\infty^2}}$, what would you say?

At ~~$M_\infty = 1$~~ Mach One, the pressures become infinite, thus there is a "sound barrier" for airplanes. We track the results so far and we see the barrier.

Q: What would you say to refute this claim?

P.G Equation is not valid at $M=1$. Not the Navier-Stokes equation when non-linear terms are dropped.

What we actually see in reality is that the curve flattens out



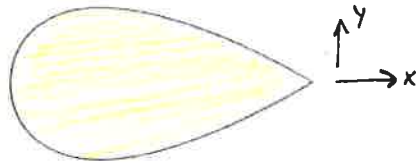
Visual Compressibility via PG

WARNING: Please don't think that these streamline accurately represent the NACA 0050 at Mach 0.90. This analysis is purely considering the visual effect of compressibility. This process occurs for thinner sections, but the transformation is not as dramatic.

WARNING: PG requires small v and w perturbations. This NACA 0050 fails that requirement.

NACA 0050
Thickness Exaggerated for effect
Mach = 0.9

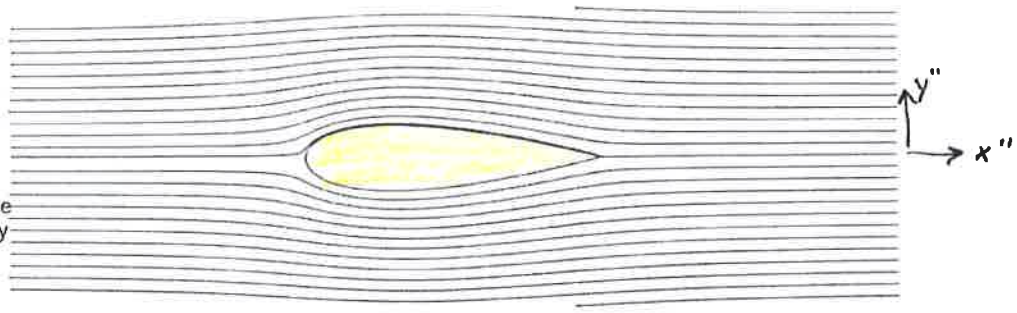
$$\beta = \sqrt{1 - M_\infty^2}$$



Transform with PG
Beta = 0.43

y" = 0.43 y
x" = x

Solve incompressible flow about effectively an NACA 0022

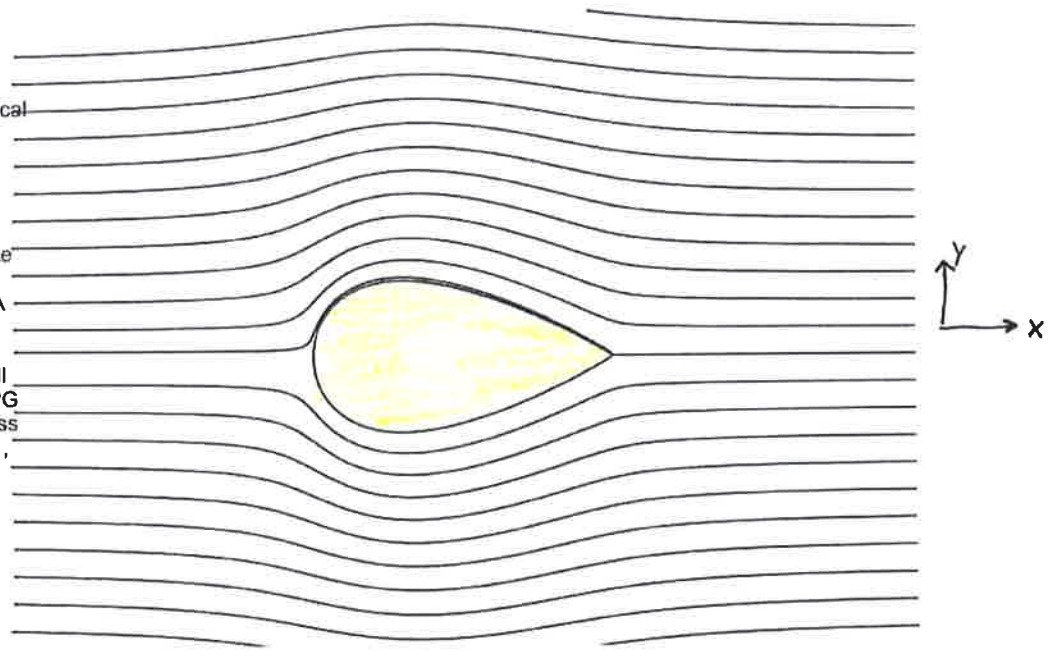


Rescale back to physical dimensions

y = 2.29 y"
x = x"

This is the PG estimate of the streamlines at Mach 0.9 for an NACA 0050.

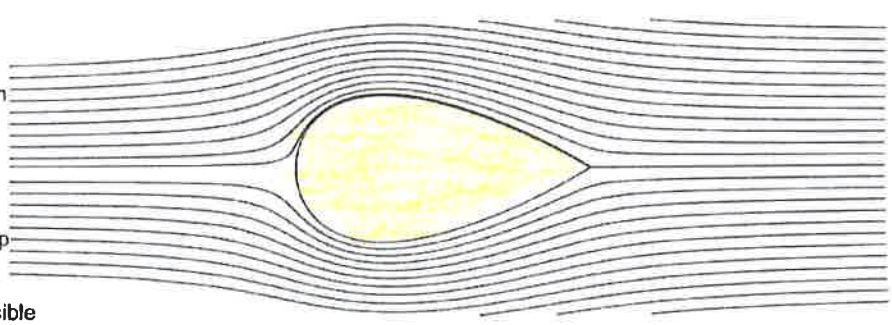
The actual solution will contain shocks. The PG transform becomes less accurate as M nears 1, since the higher order TSD approximation terms were dropped.



This is the equivalent incompressible solution for an NACA 0050.

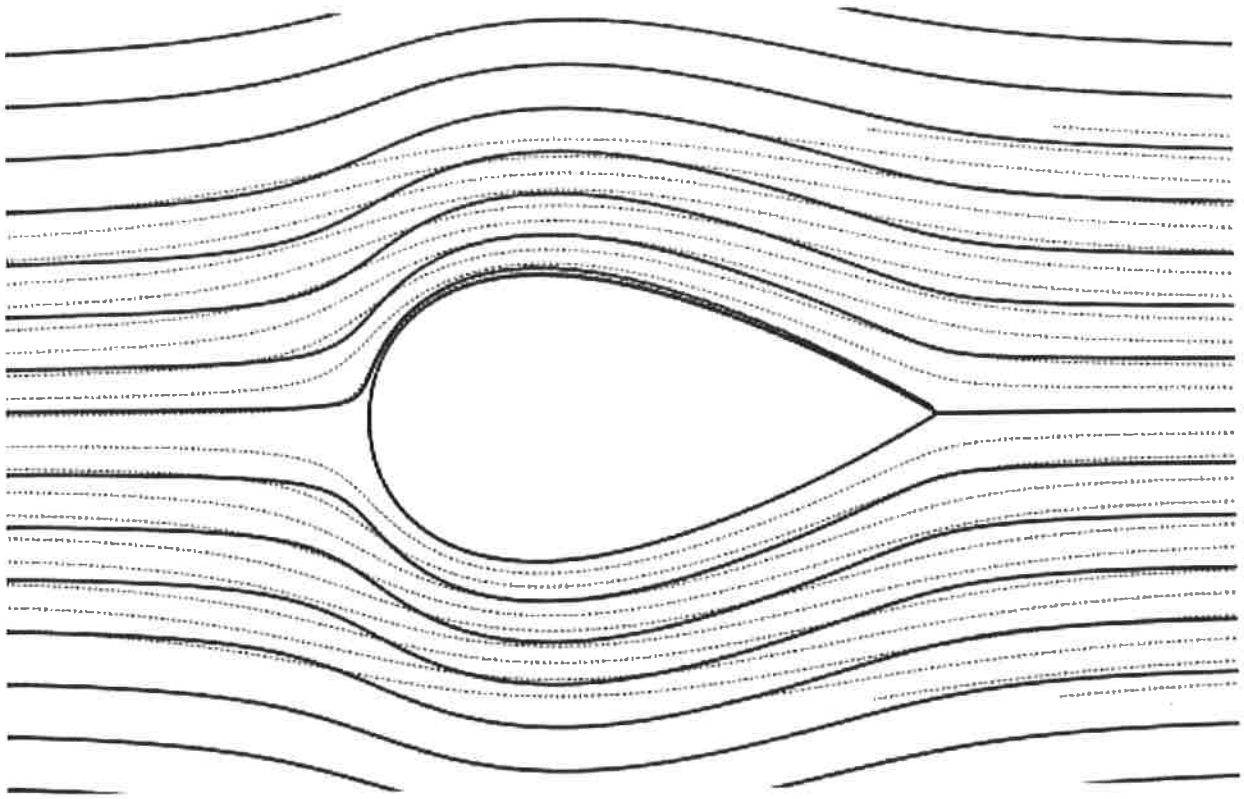
Pay attention to the streamlines.

Remember that the gap between streamlines is a visual indication of velocity for incompressible flows.



Visual comparison of incompressible (dashed line) and compressible (solid line) for exaggerated NACA 0050 case at Mach 0.9.

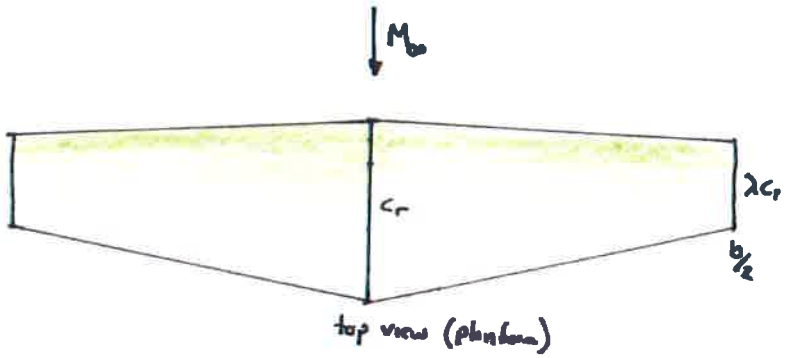
Consistent with previous analysis, streamlines in compressible flows tend to thicken and propagate outward further.



charles-oneill.com 2015

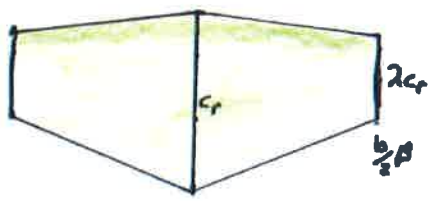
Recall our discussion of streamtubes.

Subsonic Compressible Wing



The Prandtl-Glauert equation $(1-M_\infty^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$ allows a transformation from M_∞ to $M = 0$ (incompressible).

$$\bar{x} = x$$
$$\bar{y} = \beta y \qquad \beta = \sqrt{1-M_\infty^2}$$
$$\underbrace{\bar{z}}_{\text{incomp}} = \underbrace{\beta z}_{\text{comp}}$$



$$\overline{AR} = \beta AR$$

$$\bar{\alpha} = \beta\alpha \qquad \bar{n} = \begin{pmatrix} \beta n_x \\ n_y \\ n_z \end{pmatrix}$$

From lifting line theory,

$$\bar{C}_L \approx \frac{C_{l_\alpha}}{1 + C_{l_\alpha}/\pi\overline{AR}} \cdot \bar{\alpha} \qquad \text{and} \qquad \bar{C}_{D_i} = \frac{\bar{C}_L^2}{\pi\overline{AR}\,\bar{e}}$$

Apply incomp and PG transform, and $\underbrace{\text{Göthert's rule}}$.

$$C_p = \frac{1}{\beta^2}\bar{C}_p \Rightarrow C_L = \frac{1}{\beta^2}\bar{C}_L \quad \text{and} \quad C_D = \frac{1}{\beta^2}\bar{C}_D$$

$$C_L = \frac{1}{\beta^2}\cdot\bar{C}_L = \frac{1}{\beta^2}\cdot\frac{C_{l_\alpha}}{1+C_{l_\alpha}/\pi\beta AR}\cdot\beta\alpha = \frac{C_{l_\alpha}}{\beta+\frac{C_{l_\alpha}}{\pi AR}}\alpha$$

$$C_{D_i} = \frac{1}{\beta^3}\bar{C}_{D_i} = \frac{1}{\beta^3}\frac{\bar{C}_L^2}{\pi\beta AR\,\bar{e}} = \frac{1}{\beta^2}\frac{(\beta^2 C_L^2)^2}{\pi\beta AR\,\bar{e}} = \frac{C_L^2}{\pi AR\,\bar{e}}$$

High AR wings

$$\beta \gg \cancel{\frac{C_{l_\alpha}}{\pi AR}}^{\,0} \Rightarrow C_{L_\alpha}\underset{AR\to\infty}{\approx}\frac{C_{l_\alpha}}{\beta}$$

Low AR wings

$$\beta \ll \cancel{\frac{C_{l_\alpha}}{\pi AR}}^{\,\infty} \Rightarrow \frac{C_{l_\alpha}}{\frac{C_{l_\alpha}}{\pi AR}} \Rightarrow \pi AR$$

$$C_{L_\alpha}\underset{AR\to 0}{\approx}\pi AR$$

Not a function of β!

$$\boxed{\text{Low aspect ratio wings are less influenced by compressibility.}}$$

$$\boxed{\begin{array}{l}\text{Compressibility has only a slight negative impact on induced drag for non-elliptical planforms.}\\ \qquad C_{D_i}{}_{M} \gtrsim C_{D_i}{}_{incomp}\end{array}}$$