

Lesson 25

Boundary Layer Analysis

For an in-depth study of boundary layers
and viscous flows, please consider taking:

Viscous Flow (AEM 621)

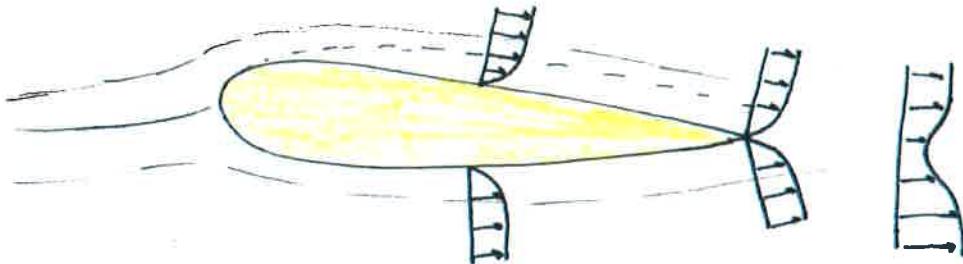
Turbulent Flow (AEM 622)

and

Flow Control (AEM 630)

Boundary Layer Analysis

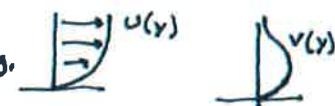
Having seen that the viscous BL (boundary layer) is a strong contributor to airfoil and wing performance, the objective is to characterize, analyze, and create models of airfoil BLs.



Approaches:

- Experimental (expensive in time + money)
- Computational (closure problem, expense, not generic)
- Inviscid (!!?!)
- BL model

Approach to modeling BL

- U, V as functions of location e.g. 
- Integrated "defects"
 δ^* displacement thickness (not unique) e.g. $\delta^* = 1 \text{ unit}$ 

Objective:

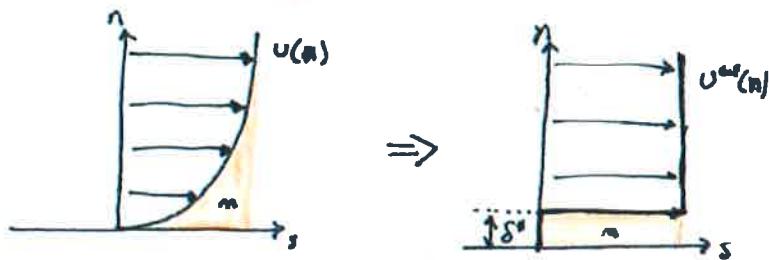
A BL model that describes the evolution of a BL property as a function of path and flow properties

$$\frac{d\theta}{ds} = f(\dots)$$

$$\frac{d\delta^*}{ds} = f(\dots)$$



Mass displacement thickness



$$\delta^* = \int_0^{n_e} \left(1 - \frac{\rho u}{\rho_e u_e}\right) dn \quad \text{and mass defect } m = \rho_e u_e \delta^*$$

$$= \int_0^{\infty} (1 - u) dn \quad (\text{incompressible}) \quad \Rightarrow \quad u = \frac{u}{u_e}$$

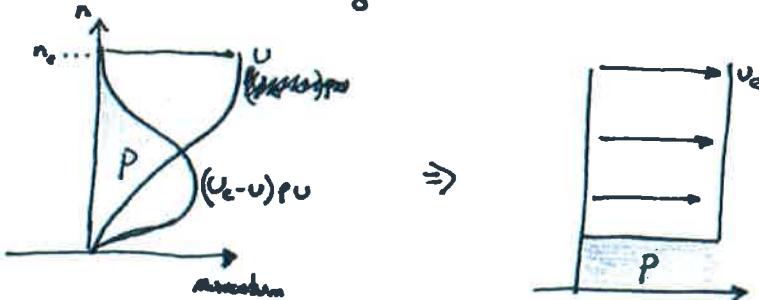
Momentum Thickness

$$\Theta = \int_0^{n_e} \left(1 - \frac{u}{u_e}\right) \frac{\rho u}{\rho_e u_e} dn = \underbrace{\int_0^{n_e} (u - u^2) dn}_{\text{incomp' only}}$$

and the momentum defect

$$P = \int_0^{n_e} (u_e - u) \rho u dn = \rho_e u_e^2 \Theta$$

! where have we seen P before?
Drag = P



Energy Thickness

$$\Theta^* = \int_0^{n_e} \left(1 - \frac{u^2}{u_e^2}\right) \frac{\rho u}{\rho_e u_e} dn$$

and the energy defect

$$K = \int_0^{n_e} \frac{1}{2} (u_e^2 - u^2) \rho u dn = \frac{1}{2} \rho_e u_e^3 \Theta^*$$

Integral Boundary Layer Relations

Take mass and momentum equations and combine to $(U_e - U) \cdot (\text{Mass eqn}) - (\text{mom eqn}) =$

$$\frac{d}{ds} (\rho_e U_e^2 \Theta) = \tau_w - \rho_e U_e \delta^* \frac{du_e}{ds}$$

$$\boxed{\frac{dp}{ds} = \tau_w + \delta^* \frac{dp}{ds}}$$

Momentum defect (i.e. drag) results from surface shear stress and an increase of pressure in the presence of a boundary layer.

The dimensionless form is

$$\frac{d\theta}{ds} = \frac{C_f}{2} - (H + 2 - M_e^2) \frac{\theta}{U_e} \frac{du_e}{ds}$$

ODE

↓
 skin
 friction
 coeff.
 ↓
 shape
 parameter
 ↓
 edge
 Mach
 #
 ↓
 pressure
 gradient
 equivalent

$$\boxed{\frac{d\theta}{ds} \approx \frac{C_f}{2} + (H + 2 - M_e^2) \frac{\theta}{U_e} \frac{dp}{ds}}$$

$$H = \frac{\delta^*}{\theta}$$

$$H_{\text{blasius BL}} \approx 2.59$$

$$H_{\text{turb}} \approx 1.4$$

Above $M_e \approx 1.8_{\text{turb}} \rightarrow 2.1_{\text{blasius}}$, the sign of $(H + 2 - M_e^2)$ switches.

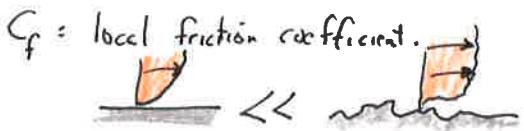
Below $M_e \approx 1.8$, $\frac{dp}{ds} > 0$ increases θ . Above, $\frac{dp}{ds} < 0$ increases θ .

See FVA Chapter 4.5 for additional information

Flight Vehicle Aerodynamics, Drile

$$\frac{d\theta}{ds} = \frac{C_f}{2} + (H + 2 - M_c^2) \frac{\theta}{\rho_e U_e^2} \frac{dp}{ds}$$

θ is a measure of the boundary layer thickness and impact on drag. "Momentum thickness"



H = shape parameter
 ≈ 1.4 turb

$\frac{dp}{ds}$ = pressure gradient along streamline at surface

Q: Given zero pressure gradient, how does θ evolve?

θ is a measure of the momentum thickness (i.e. drag up to that point)

$$\frac{d\theta}{ds} = \frac{C_f}{2} + \dots \theta \dots \frac{dp}{ds} = 0$$

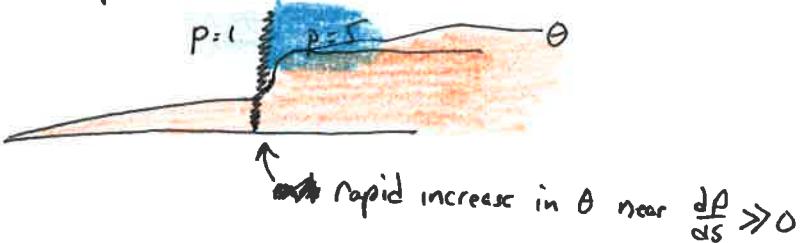
θ

Drag depends on surface friction only when $dp/ds = 0$

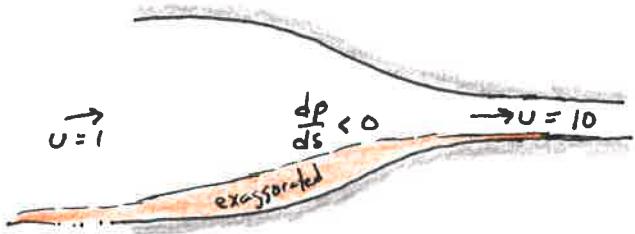
Q: Given a strong adverse pressure gradient ($\frac{dp}{ds} \gg 0$), how does θ evolve?

$$\frac{d\theta}{ds} = \frac{C_f}{2} + (H + 2 - M_c^2) \frac{\theta}{\rho_e U_e^2} \frac{dp}{ds} \xrightarrow{\text{very large}} \theta$$

θ increases rapidly. We see this aft of shocks.



Q: Give a wind tunnel contraction, how does θ evolve?



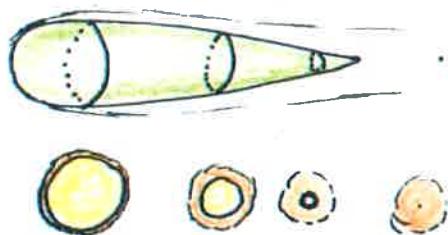
Bernoulli's eqn.

$$\frac{d\theta}{ds} = \frac{C_f}{2} + (H + 2 - M_c^2) \frac{\theta}{\rho_e U_e^2} \frac{dp}{ds}$$

if $\frac{dp}{ds} < -\frac{C_f}{2} \left(\frac{1}{M_c^2} \right) \left(\frac{\rho_e U_e^2}{\theta} \right)$ then the BL thins!

$$\frac{1}{2} \rho V_1^2 + P_1 = \frac{1}{2} \rho V_2^2 + P_2 \Rightarrow P_2 - P_1 = \frac{1}{2} \rho (V_1^2 - V_2^2) \Rightarrow P_2 \ll P_1 \quad \frac{dp}{ds} < 0$$

Axissymmetrical Bodies



b = perimeter at a section

- BL behavior for 3D shapes differs from a 2D shape because of flow divergence resulting from the area change

"Over the rear of the body, the momentum thickness $\theta(s)$ increases faster than in 2D, due to the viscous fluid flowing onto a progressively smaller perimeter" (FVA Fig 4.17)

3D:

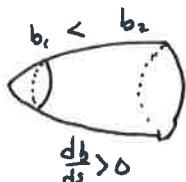
$$\frac{d\theta}{ds} = \frac{C_f}{2} - \left(H + 2 - M_e^2 \right) \frac{\theta}{U_e} \frac{du_e}{ds} - \underline{\frac{\theta}{b} \frac{db}{ds}}$$

$\frac{db}{ds}$ is the rate of change of the perimeter.

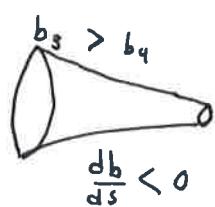
2D:

$$\frac{d\theta}{ds} = \frac{C_f}{2} - \left(H + 2 - M_e^2 \right) \frac{\theta}{U_e} \frac{du_e}{ds}$$

- When the perimeter is growing (i.e. forward portion) $\frac{db}{ds} > 0$ and $\frac{d\theta}{ds}$ grows more slowly than 2D



- When the perimeter is shrinking (aft portion), $\frac{db}{ds} < 0$ and $\frac{d\theta}{ds}$ grows faster than 2D.

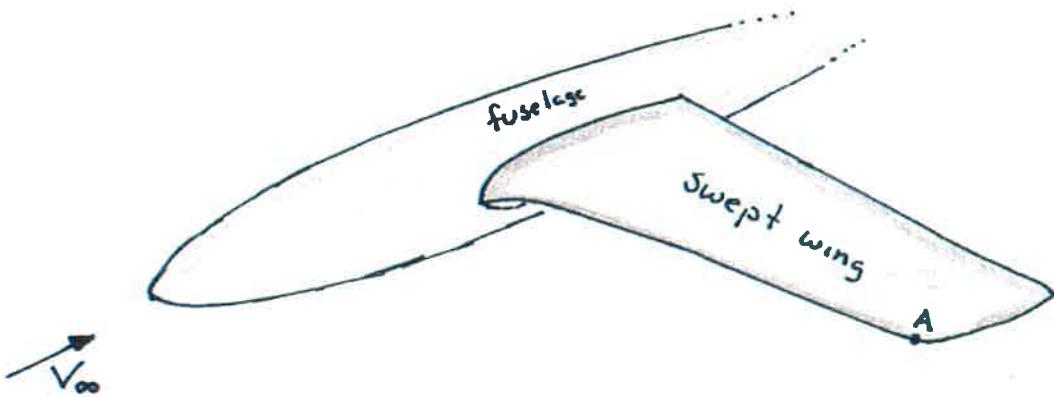


Aside.

Have you ever wondered why the BL in a wind tunnel is relatively ~~thick~~ thin?

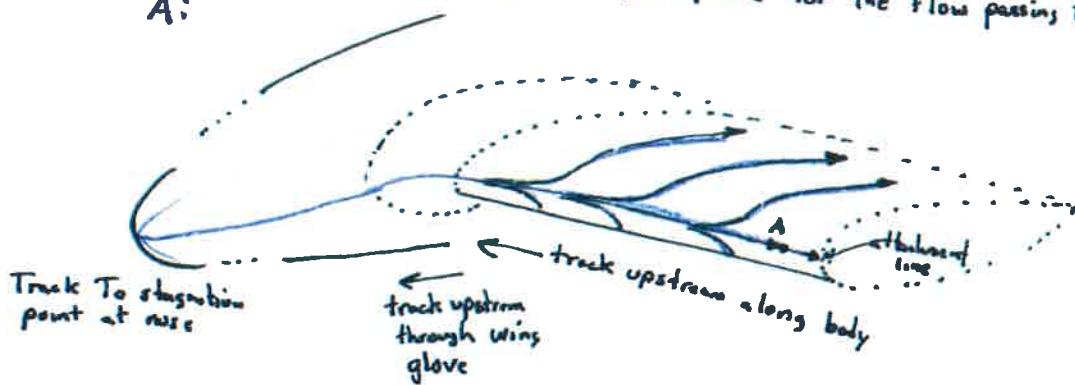
$$\frac{dp}{ds} \text{ and } \frac{db}{ds}$$

Crossflow Convergence and Divergence.



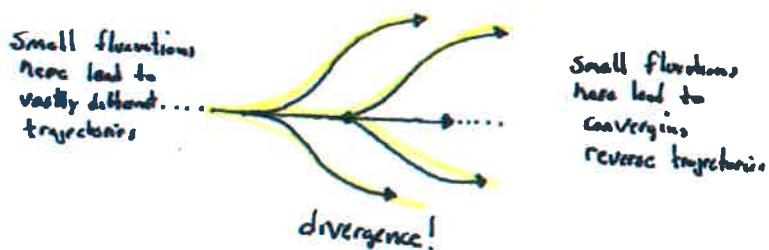
Q: Where is the stagnation point/line for the flow passing through point A?

A:



Q: Boundary Layer thickness depends on distance "s" from start/stagnation pt. So, how thick is the BL at A? It should be very thick after traveling that distance, right?

A: Very thin! Divergence along the attachment line keeps the BL thin.



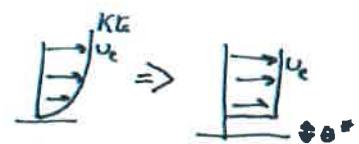
Recall the 3D axisymmetric term $-\frac{\partial}{\partial s} \frac{db}{ds}$ and reform into a velocity divergence term.

Q: What about the TE?

Integral Boundary Layer

Extra Notes (not included on exam)

$$\theta^* = \int_0^{\infty} \left(1 - \frac{u^2}{U_e^2}\right) \frac{p_u}{p_e U_e} dn \quad \text{Kinetic energy thickness}$$



How does θ^* evolve with s ?

$$\frac{d}{ds} \left(\underbrace{\frac{1}{2} p_e U_e^3 \theta^*}_K \right) = \underbrace{D}_{\text{Dissipation Integral}} - \underbrace{p_e U_e^2 \delta^{**} \frac{du}{ds}}_{\substack{\text{density variation} \\ \text{in pressure gradient}}} = \frac{dK}{ds}$$

Compare with vorticity generation when $\nabla p \times \nabla p \neq 0$ previously in class.

when the density flux thickness is $\delta^{**} = \int_0^{\infty} \left(1 - \frac{p}{p_e}\right) \frac{u}{U_e} dn$

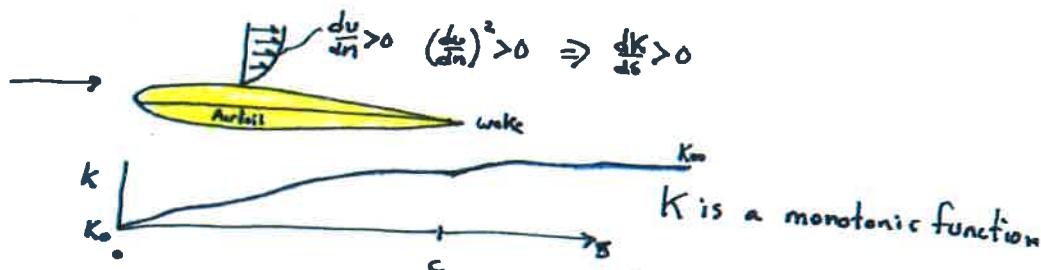
$$D = \int_0^{\infty} T \frac{du}{dn} dn = \int_0^{\infty} (\mu + \mu_T) \left(\frac{du}{dn}\right)^2 dn \quad \text{when } \sim \text{Newtonian-Boersmaeq viscosity flow}$$

Dissipation is a positive function of $\frac{du}{dn}$

$$T \approx (\mu + \mu_T) \frac{du}{dn}$$

For incompressible flows (or flows where density in the BL changes only slightly),

$$\frac{dK}{ds} = D \geq 0 \quad \text{on both the airfoil and in the wake.}$$



Drag Power Balance

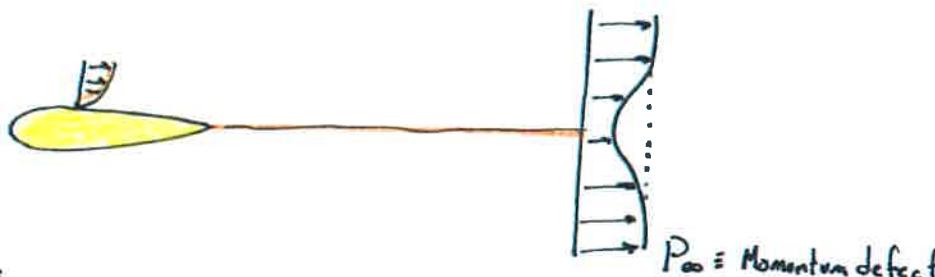
$$D' V_\infty = K_\infty = \int_{-\infty}^{\infty} D ds$$

$$D' = \underbrace{\int_0^{\infty} \int_0^{\infty}}_{\substack{\text{volume} \\ \text{integral}}} (\mu + \mu_T) \left(\frac{du}{dn}\right)^2 dn$$

Remember that this uses the Thin Boundary Layer equation (TSL) as a starting point.

Extra Notes (not included on exam)

Drag Comparison of Momentum Deficit/Detect and Dissipation / Kinetic Energy Deficit



Momentum:

$$\frac{\text{Drag}}{\text{span}} = D' = P_\infty \quad \text{where} \quad \frac{dp}{ds} \approx \tau_w + \delta^* \frac{du}{ds}$$

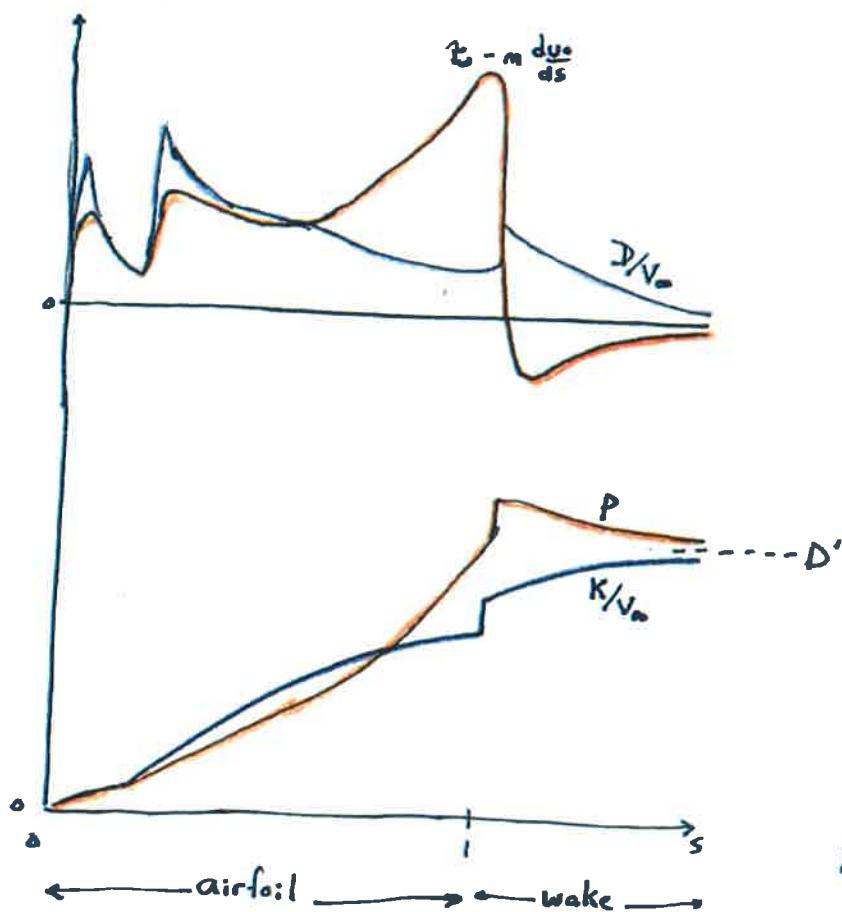
$$= \tau_w + \rho_e u_e \delta^* \frac{du_e}{ds} = \tau_w + \cancel{\rho_e u_e} \frac{du_e}{ds}$$

Energy:

$$D' = \frac{1}{V_\infty} \int_0^\infty \int_{\infty}^{\infty} (\nu + \mu_s) \left(\frac{du}{ds} \right)^2 dn$$

$\underbrace{\hspace{100pt}}$
This is obviously strictly positive!

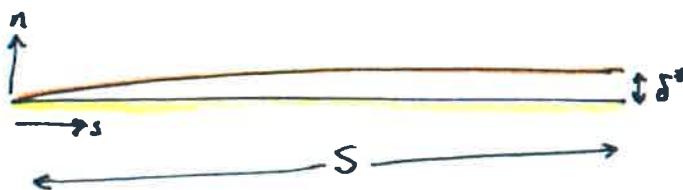
strictly positive?
Separation gives $\tau_w < 0$
Magnitude of τ_w vs. $\frac{du}{ds}$ due?



FVA Fig 4.10

Extra Notes

Thin Shear Layer (TSL)



$$\frac{\delta^*}{S} \ll 1$$

For higher Re airfoils, the BL is thin. Thus,

$$V \ll U$$

$$\frac{du}{ds} \ll \frac{du}{dn} \Rightarrow \bar{\gamma} \approx \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}$$

$$\frac{\partial p}{\partial n} \approx 0$$

$$\frac{dp}{ds} \approx - \rho_e U_e \frac{du_e}{ds}$$

slowing the flow
increases $\frac{dp}{ds}$

The TSL governing (resulting from above and N-S eqns) are:

$$\frac{\partial(\rho u)}{\partial s} + \frac{\partial(\rho v)}{\partial n} = 0 \quad \text{mass}$$

$$\rho u \frac{du}{ds} + \rho v \frac{du}{dn} = \underbrace{\rho_e U_e \frac{du_e}{ds}}_{\text{pressure law}} + \frac{\partial \gamma}{\partial n} \quad \text{momentum}$$

$$\tau = (\mu + \mu_t) \frac{du}{dn}$$

μ = turbulent viscosity

Boussinesq viscosity approx

BCs

$$u(n=0) = v(n=0) = 0$$

$$u(n=n_e) = U_e$$

3 unknowns ($\rho u, \rho v, \tau$) and 3 equations

need γ for τ if not incomp.

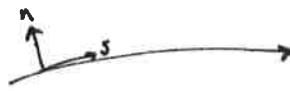
Some CFD solvers for aircraft prototyping/design use this approximation for fast viscous solutions.

e.g. NSU3D

Extra Notes

Pressure Gradients and Shear Stress Gradients

Along a streamline, $v=0$.



TSL becomes

$$\underbrace{\rho U \frac{du}{ds}}_{\text{Velocity in BL}} = \underbrace{\rho_e U_e \frac{du_e}{ds}}_{\text{inviscid flow terms}} + \underbrace{\frac{dT}{dn}}_{\text{Shear "pressure"}}$$

\approx

$$\frac{du}{ds} = \frac{\rho_e U_e}{\rho U} \frac{du_e}{ds} + \frac{1}{\rho U} \frac{dT}{dn}$$

$$\underbrace{\frac{du}{ds} \approx -\frac{1}{\rho U} \frac{dp}{ds} + \frac{1}{\rho U} \frac{dT}{dn}}_{\substack{\text{Spatial rate of change of velocity}}} \approx -\frac{1}{\rho U} \frac{dp}{ds} + \frac{1}{\rho U} (N + N_t) \frac{\partial^2 u}{\partial n^2}$$

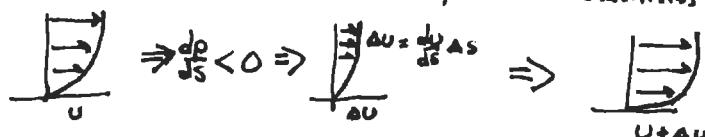
Negative pressure gradient speeds up u vel.

Parabolic PDE term "diffusion of u "

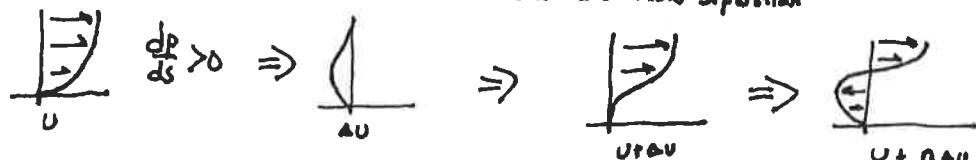
Magnitude scaled by $\frac{1}{\rho U}$. Fast flows are slower to change for a given $\frac{dp}{ds}$ or $\frac{dT}{dn}$

Conclusions

- Favorable pressure gradients fill out BL velocity and are stabilizing



- Adverse pressure gradients tend to cause flow reversal and flow separation



- Viscosity diffuses a velocity deficit into the flow in the normal direction

