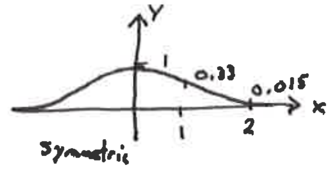
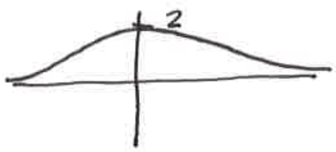
PS 3 Hints/Solution

1) $\underbrace{V_\infty - u}_{\text{deficit}} = \underbrace{A\, V_\infty}_{\text{strength}}\ \underbrace{e^{-By^2}}_{\text{shape}}$

• $y = e^{-x^2}$

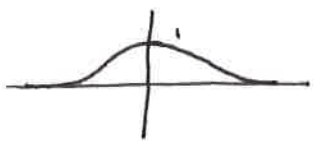


• $y = 2e^{-x^2}$



Double the magnitude.

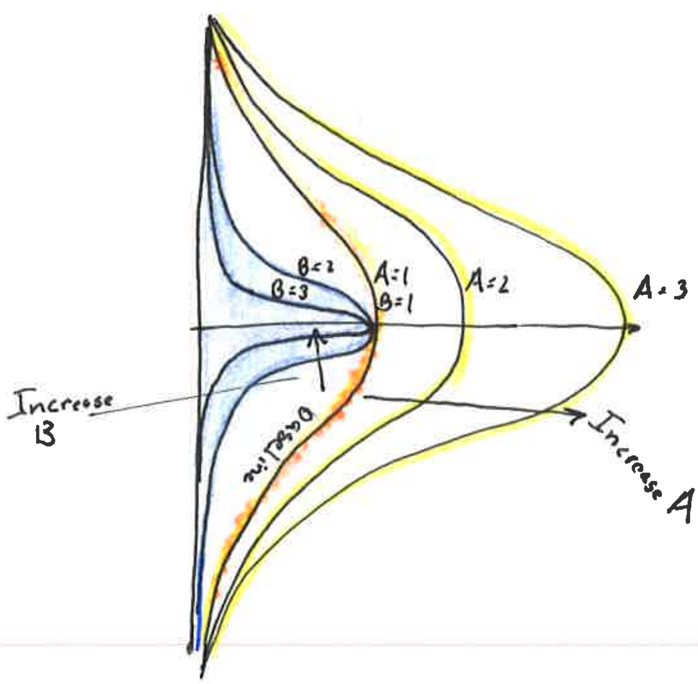
• $y = e^{-2x^2}$



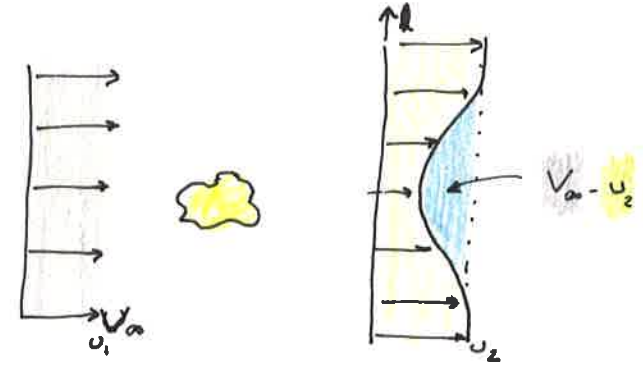
For the same value of x, $2x^2$ is larger and thus $-2x^2$ is smaller and e^{-2x^2} is smaller.

So for the deficit,

$$\frac{V_\infty - u}{V_\infty} = A\, e^{-By^2}$$



$$C_d = \frac{D'}{\frac{1}{2}\rho V_\infty^2 c}$$



$$D' = \int \rho_2 u_2 \left(u_1 - u_2\right) d\ell$$

$$V_\infty - u_2 = A V_\infty e^{-By^2} \Rightarrow u_2 = V_\infty\left(1 - Ae^{-By^2}\right)$$

$$= \int_{-\infty}^{\infty} \rho_2 V_\infty \left(1 - Ae^{-By^2}\right)\left(AV_\infty e^{-By^2}\right) d\ell$$

$$= \int_{-\infty}^{\infty} \rho_2 V_\infty^2 \left(1 - Ae^{-By^2}\right)\left(Ae^{-By^2}\right) d\ell$$

The defect is measured far downstream, such that ρ_2 is constant.

$$= \rho_\infty V_\infty^2 \int_{-\infty}^{\infty} \left(Ae^{-By^2} - A^2 e^{-By^2} e^{-By^2}\right) d\ell$$

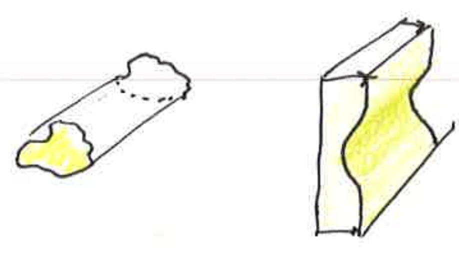
$$= \rho_\infty V_\infty^2 \int_{-\infty}^{\infty} \left(Ae^{-By^2} - A^2 e^{-2By^2}\right) d\ell$$

- But the function e^{-By^2} is symetric, so $\int_{-\infty}^{\infty} e^{-By^2} = 2\int_0^{\infty} e^{-By^2}$ this puts it in the form given in the hint!

$$\int_0^{\infty} e^{(-By^2)} = \frac{1}{2}\sqrt{\frac{\pi}{B}}\, \cancel{\mathrm{Erf}(\infty)}^{\,1} = \frac{1}{2}\sqrt{\frac{\pi}{B}}$$

$$= \rho_\infty V_\infty^2\, 2\Bigg[\underbrace{\int_0^{\infty} Ae^{-By^2} d\ell}_{A\frac{1}{2}\sqrt{\frac{\pi}{B}}} + \underbrace{\int_0^{\infty} -A^2 e^{-2By^2} d\ell}_{-A^2\frac{1}{2}\sqrt{\frac{\pi}{2B}}} = \rho_\infty V_\infty^2 \left(A\sqrt{\frac{\pi}{B}} - A^2\sqrt{\frac{\pi}{2B}}\right)$$

Notice that this is a drag per unit span



=

$$C_d = \frac{D'}{\frac{1}{2}\rho V_\infty^2 c}$$

$$= \frac{\rho_\infty V_\infty^2 \left(A\sqrt{\frac{\pi}{B}} - A^2\sqrt{\frac{\pi}{2B}}\right)}{\frac{1}{2}\rho V_\infty^2 c}$$

$$= \frac{2A\sqrt{\frac{\pi}{B}} - 2A^2\sqrt{\frac{\pi}{2B}}}{c}$$

plug in #s. $A = 0.5$ $B = 0.1$

$$C_d = \frac{2 \cdot 0.5\sqrt{\frac{\pi}{0.1}} - 2 \cdot 0.5^2 \cdot \sqrt{\frac{\pi}{2 \cdot 0.1}}}{c}$$

$$\boxed{\frac{C_d}{c} = 3.62}$$

Q: Given that the drag is constant, how must A and B vary with respect to each other. (i.e. $A = f(B)$)

$$2A\sqrt{\frac{\pi}{B}} - 2A^2\sqrt{\frac{\pi}{2B}} = \text{Constant} = \frac{C_d}{c}$$

Solve for A (div by 2A)

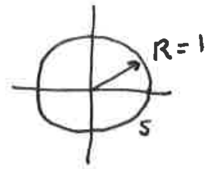
$$\cancel{A}\ \sqrt{\frac{\pi}{B}} - A\sqrt{\frac{\pi}{2B}} = \frac{C_d}{c} \quad \Rightarrow A\sqrt{\frac{\pi}{2B}} = \sqrt{\frac{\pi}{B}} - \frac{C_d}{c}$$

$$A = \sqrt{\frac{2B}{\pi}}\sqrt{\frac{\pi}{B}} - \sqrt{\frac{2B}{\pi}}\frac{C_d}{c} = \boxed{\sqrt{2} - \sqrt{\frac{2B}{\pi}}\frac{C_d}{c} = A}$$

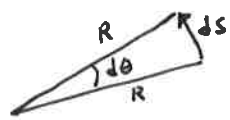
4) $U_r = \frac{\Lambda}{2\pi r}$ $\quad U_\theta = \frac{\Gamma_{\text{Vortex}}}{2\pi r}$

- Circulation

$$\Gamma = -\oint V \cdot ds$$



this means an integral over a path/curve called s.
We can convert this to a regular integral by parameterizing ds.



$$ds = R\,d\theta$$

$V \cdot dS$ is the velocity component along dS. Since dS is along a unit circle, only U_θ will contribute.

$$\Gamma = -\int_0^{2\pi} U_\theta R\,d\theta = -\int_0^{2\pi} \frac{\Gamma_{\text{Vortex}}}{2\pi R} R\,d\theta$$

$$= -\int_0^{2\pi} \frac{\Gamma_{\text{Vortex}}}{2\pi}\,d\theta = -\frac{\Gamma_{\text{Vortex}}}{2\pi} \int_0^{2\pi} d\theta \quad 2\pi - 0 = 2\pi$$

$$\Gamma = -\Gamma_{\text{Vortex}}$$

We define the vortex strength by the resulting circulation (magnitude).

Divergence.

$$\sigma = \nabla \cdot V$$

In cylindrical/polar coordinates: $\operatorname{div} F = \frac{1}{r}\frac{\partial}{\partial r}(r F_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}$

$$\sigma = \frac{1}{r}\frac{\partial}{\partial r}(r V_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\Lambda}{2\pi r}\right) + \frac{1}{r}\cancelto{0}{\frac{\partial\left(\frac{\Gamma}{2\pi r}\right)}{\partial \theta}}$$

$$= \frac{1}{2}\cancelto{0}{\frac{\partial}{\partial r}\left(\frac{\Lambda}{2\pi}\right)}$$

$$= 0 \quad \checkmark$$

Well, except we ignored $r=0$ and canceled $r \frac{\Lambda}{2\pi r}$ when this is actually undefined at $r=0$

Vorticity

$$\omega = \nabla \times V$$

$$= \frac{1}{r}\left(\frac{\partial}{\partial r}(r V_\theta) - \frac{\partial V_r}{\partial \theta}\right)$$

$$= \frac{1}{r}\left(\cancelto{0}{\frac{\partial}{\partial r}\left(r\frac{\Gamma}{2\pi r}\right)} - \cancelto{0}{\frac{\partial}{\partial \phi}\left(\frac{\Lambda}{2\pi r}\right)}\right)$$

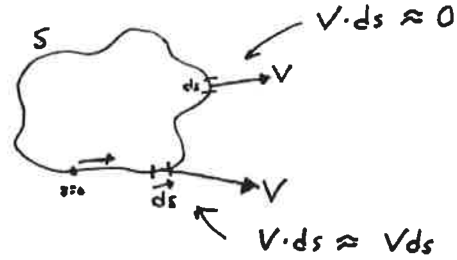
$= 0$ except at $r=0$

Hint PS #3 problem 4.

Computing Circulation

Concept
$$\Gamma = -\oint V \cdot ds$$

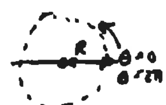
↑ This is called a contour integral "around" a closed curve



Convert contour integral to reguler Integral.

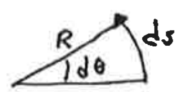
- pick a function describing the closed curve
  (this is why I gave you the simple unit circle in the $\left\{\begin{matrix}PS\\HW\end{matrix}\right.$!)

  unit circle $\begin{cases} r = R \\ \theta = 0 \ldots 2\pi \end{cases}$



- Determine ds for your closed curve

  $$ds = R\,d\theta$$



- Determine $V \cdot ds$ for our unit circle.

  $$V \cdot ds = V_\theta ds$$

- plug into contour integral

$$\Gamma = -\oint V \cdot ds = -\oint V_\theta ds$$

$$= -\int_0^{2\pi} V_\theta \, R\,d\theta = -\int_0^{2\pi} \frac{\Gamma}{2\pi R} R\,d\theta = \ldots .$$