

Review of Vector Calculus

For Cartesian Coordinates

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

$$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

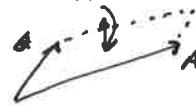
Dot product (inner product)

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z = AB \cos \theta$$

Cross product

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y) \hat{i} - (A_x B_z - A_z B_x) \hat{j} + (A_x B_y - A_y B_x) \hat{k}$$

= Area of parallelogram of sides \vec{A} and \vec{B}
 $\vec{A} \times \vec{B}$ is normal direction



Derivatives of vectors

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt} \hat{i} + \frac{dA_y}{dt} \hat{j} + \frac{dA_z}{dt} \hat{k}$$

also

$\frac{d}{dt}(a\vec{A}) = \frac{da}{dt}\vec{A} + a \frac{d\vec{A}}{dt}$	}	Same algebraic operations as scalars
$\frac{d}{dt}(\vec{A} \cdot \vec{B}) = \vec{A} \cdot \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \cdot \vec{B}$		
$\frac{d}{dt}(\vec{A} \times \vec{B}) = \vec{A} \times \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \times \vec{B}$		

Field

$$\vec{F} = f(x, y, z) \quad \text{or} \quad \text{Temp} = T(x, y, z)$$

Gradient ("del")

$$\nabla \phi = \text{grad } \phi$$

- in 3D cartesian $\nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$
- gives direction vector of largest magnitude increase (slope)
- perpendicular to surface of constant ϕ

Directional Derivative .

$$\frac{d\phi}{ds} = \nabla \phi \cdot \hat{s} \quad \text{where } \hat{s} \text{ is a unit vector}$$

Vector Calc (continued)

Divergence

$$\nabla \cdot A = \operatorname{div} A = \frac{dA_x}{dx} + \frac{dA_y}{dy} + \frac{dA_z}{dz}$$

if $\nabla \cdot A = 0$, A is solenoidal

Curl

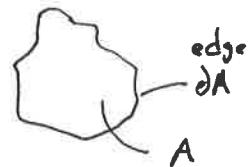
$$\nabla \times A = \operatorname{curl} A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} \quad \text{if } \nabla \times V = 0, V \text{ is irrotational}$$

Laplacian

$$\nabla^2 A = \nabla \cdot \nabla A = \operatorname{div} \operatorname{grad} A = \frac{d^2 A}{dx^2} + \frac{d^2 A}{dy^2} + \frac{d^2 A}{dz^2}$$

Green's theorem

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_{\partial A} P dx + Q dy$$



Divergence theorem

$$\iiint_A \operatorname{div} V dA = \iint_{\partial A} V \cdot n d\hat{A}$$

Stokes' theorem

$$\iint_{\sigma} (\operatorname{curl} V) \cdot n d\sigma = \int_{\partial\sigma} V \cdot dr$$

Divergence of curl

$$\nabla \cdot (\nabla \times V) = \operatorname{div} (\operatorname{curl} V) = 0$$

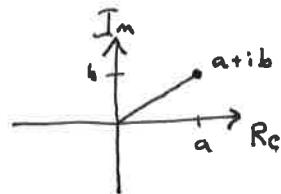
Curl of curl

$$\nabla \times (\nabla \times V) = \nabla(\nabla \cdot V) - \nabla^2 V$$

Review of Complex #'s

Number with Real and Imaginary Components

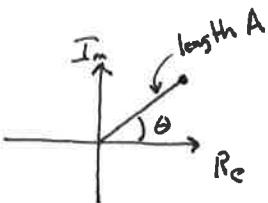
$$z = a + ib$$



polar form

$$z = Ae^{i\theta}$$

↑ modulus ↑ argument



Conversion

$$\begin{aligned} A &= \sqrt{a^2 + b^2} \\ &= |z| \end{aligned}$$

$$\theta = \tan^{-1} \frac{y}{x}$$

Conjugate

$$\bar{z} = x - iy = re^{-i\theta} \quad \text{when } z = x + iy = re^{i\theta}$$

identities:

$$|z| = |\bar{z}|, \quad \arg \bar{z} = -\arg z, \quad |z| = \sqrt{z\bar{z}}$$

$$\bar{\bar{z}} = z, \quad \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

Operations

Add $z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = (x_1 + x_2) + i(y_1 + y_2)$

Mult

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

Power

$$z^n = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta)$$

roots

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\frac{\theta}{n}}$$

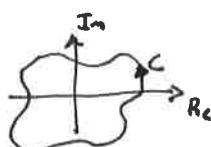
ex: $\sqrt{-1} = \sqrt{1} e^{i\frac{\pi}{2}} \Rightarrow \sqrt{-1} = \frac{\pm 1}{\sqrt{2}} + i \frac{\pm 1}{\sqrt{2}}$

$\sqrt[4]{-1} = \sqrt[4]{1} e^{i\frac{\pi}{4}}, \quad \sqrt[4]{-1} = \sqrt[4]{1} e^{i\frac{5\pi}{4}}$

$\sqrt[4]{-1} = \frac{1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}$

Cauchy Integral formula

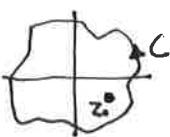
- $f(z)$ analytic everywhere $\Rightarrow \int_C f(z) dz = 0$



Analytic \equiv derivative exists at a point

- $f(z)$ analytic except at z_k inside C

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^{\infty} \text{Residue}(z_k)$$



Refer to a complex variable source for why.

$$\left| \int_C f(z) dz = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} dw \right|$$

$$\text{Residue}(z_k) = \frac{1}{2\pi i} \int_C f(z) dz$$