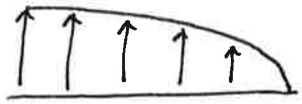
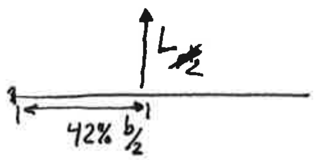
# Comparison of Elliptical Loading and Cosine Approximation

## Elliptical

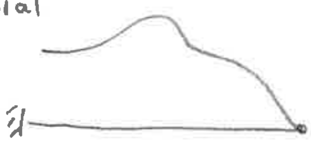


$$L' = \frac{L}{b}\frac{4}{\pi}\sqrt{1-\left(\frac{2y}{b}\right)^2}$$

Equivalent Load




## Polynomial



$$L' = A + By + Cy^2 + Dy^3$$

## Fourier (as seen in AEM 313

$L'$ = cosine terms

## Numerical / Experimental

| $y$ | $L'$ |
|---|---|
| ⋮ | ⋮ |

$$V_0 = -\int_0^{b/2} L'\,dy = \text{Numerically, Integrate } L'$$

$$M_0 = \int_0^{b/2} L'y\,dy$$

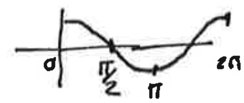
## Cosine

$$L' = A\cos(cy)$$

Boundary Conditions

$$L'(y=b/2) = 0 \quad \text{(Aerodynamics)}$$

$$L' = A\cos\underbrace{\left(c\,b/2\right)}_{\cos(\pi/2)=0} = 0$$




$$c\,b/2 = \pi/2 \Rightarrow c = \pi/b$$

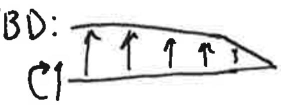
And the total force is $L/2$

$$\frac{L}{2} = \int_0^{b/2} L'\,dy = \int_0^{b/2} A\cos\left(\frac{\pi}{b}y\right)dy$$

$$\frac{L}{2} = \frac{b}{\pi}A \Rightarrow A = \frac{L\pi}{2b}$$

Total: $$L' = \frac{L\pi}{2b}\cos\left(\frac{\pi}{b}y\right)$$

FBD:



$$V_0 + \int_0^{b/2} L'\,dy = 0$$

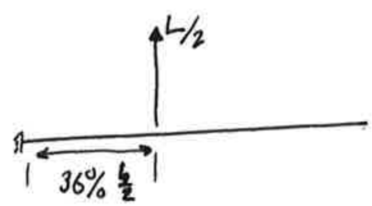
$$V_0 = -L$$

$$M_0 - \int_0^{b/2} L'y\,dy = 0$$

$$M_0 = b^2\left(\frac{1}{2\pi} - \frac{1}{\pi^2}\right)\frac{L\pi}{2b}$$

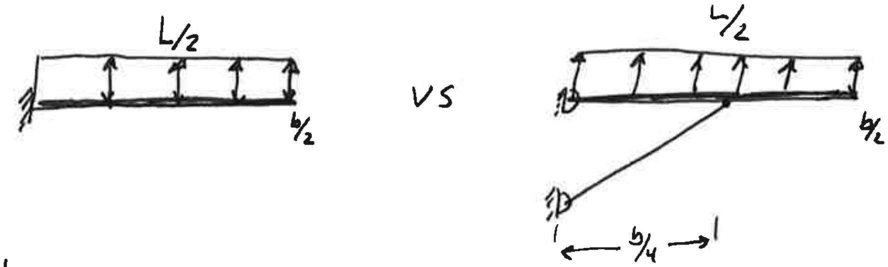
$$= \frac{L}{2}\frac{b}{2}\,2\pi\left(\frac{1}{2\pi} - \frac{1}{\pi^2}\right) = 0.36\,\frac{L}{2}\frac{b}{2}$$

Shear + Moment

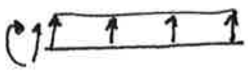



Not conservative. Root moment too low

Ex: Compare the root and maximum bending moment of a cantilever and strut braced wing. Use a uniform loading

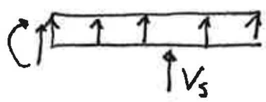



Cantilever:



$V_0 = -\frac{L}{2}$ $\qquad M_0 = -\frac{L}{2} \cdot \frac{b}{4} = \frac{-Lb}{8}$

Strut:



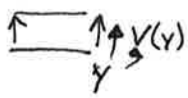

$\Sigma F_z = 0 = V_0 + \frac{L}{2} + V_s$

$\Sigma M = 0 = \cancel{M_0}^{0} + \frac{L}{2} \cdot \frac{b}{4} + V_s \cdot \frac{b}{4} \Rightarrow V_s = -L/2$

subst' into $\Sigma F_z$

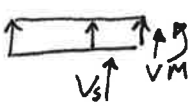
$V_0 = \frac{L}{2} - \frac{L}{2} = 0$ and $M_0 = 0$

Shear + Moment




$V(y) + \frac{L}{2}\frac{2}{b}y = 0 \Rightarrow V_y = -\frac{L}{b}y$

$-M(y) - \int_0^y \left(\frac{L}{2}\frac{2}{b}\right) y\,dy = 0 \Rightarrow +M(y) = \frac{-L}{b}\frac{1}{2}y^2$

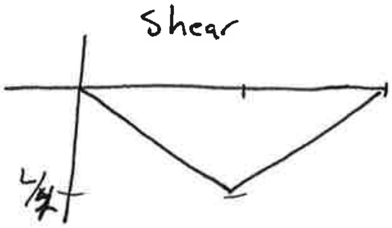


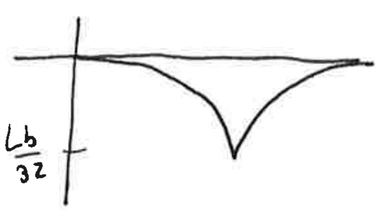

$V(y) + V_s + \frac{L}{2}\frac{2}{b}y = 0 \qquad V(y) = -V_s - \frac{L}{b}y = \frac{L}{2} - \frac{L}{b}y$

$-M(y) - V_s\frac{b}{4} - \int_0^y \left(\frac{L}{2}\frac{2}{b}\right) \cancel{y}\,dy = 0$

$M(y) = -\frac{V_s b}{4} - \frac{L}{b}\frac{1}{2}y^2 = \frac{Lb}{8} - \frac{1}{2}\frac{L}{b}y^2$

Shear




Pro:

$\boxed{\text{Shear is half} \quad \text{Moment is } 1/4}$

Con:

Wing has compressive load on inboard section from drag of strut

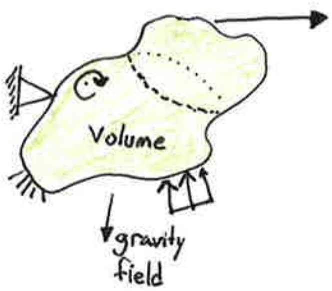
# Aerospace Structural Design

3 critical requirements .... Given a set of loads

- Stress $\sigma_{ij}$
- Deformation $\epsilon_{ij}$ and $\delta$
- Stability $\frac{d()}{dt} \xrightarrow[\lim]{} \infty$

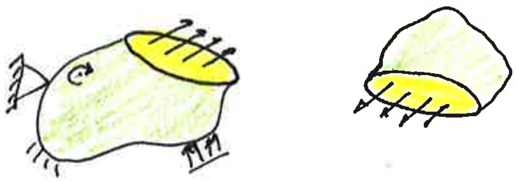
} Need to know stress and strain to evaluate failure criteria

## Stress



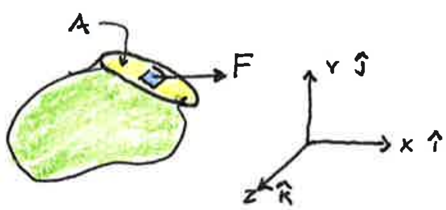

Subject to an arbitrary body/structure to loads, all physical objects will deform and create internal forces/loadings/stress fields.

If we cut the body with an arbitrary plane, there will be internal reactions

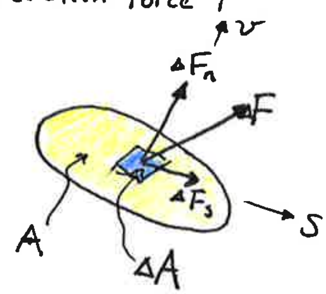


The removed piece will have equal but opposite reactions due to Newton's law.

At a small surface area on the cut, "A" there is an internal reaction force F




which can be resolved into coordinate forces or in terms of normal and tangential forces on the cut, where $\nu$ is the normal direction




A surface traction vector is the ~~limit~~ force per unit area in the limit in a direction

$$T(\nu) = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \frac{dF}{dA}$$

Also write in terms of coordinate

$$T(\nu) = T_x \hat{i} + T_y \hat{j} + T_z \hat{k} = T$$

Given the plane cut's normal vector $\nu$:

The component of A in the x direction is.

$$A_x = A\nu_x$$

and

$$A_y = A\nu_y$$

$$A_z = A\nu_z$$

# Normal and Shear Stress

Normal $\sigma_n = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A} = \frac{dF_n}{dA} = \underbrace{\boxed{T \cdot \nu}}_{\text{"pressure in a direction"}}$

Remember that the dot product $\bullet$ takes the component of a vector in a particular direction. In this case $T$ in the normal direction $\nu$.

Shear

$$\sigma_s = \lim_{\Delta A \to 0} \frac{\Delta F_s}{\Delta A} = \frac{dF_s}{dA}$$

but $\Delta F = \Delta F_n \nu + \Delta F_s s$ or $\underset{\text{vector}}{dF} = \underset{\text{scalar}}{dF_n}\, \underset{\text{vector}}{\nu} + \underset{\text{scalar}}{dF_s}\, \underset{\text{vector}}{s}$

$\sigma_s = \frac{dF_s}{dA}$ $\qquad\qquad$ $dF_s s = dF - dF_n \nu$

But we can't divide by a vector $s$ to get only $dF_s$. Rather, we will compute $\sigma_s s$

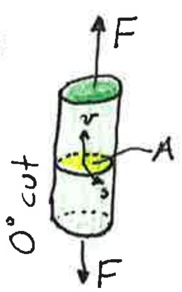
$$\sigma_s s = \frac{dF_s}{dA} s = \frac{1}{dA}(dF - dF_n \nu) s = \Big(\underbrace{\frac{dF}{dA}}_{T} - \underbrace{\frac{dF_n}{dA}}_{T\cdot\nu} \nu\Big) s$$

So the absolute value of $\sigma_s$ is

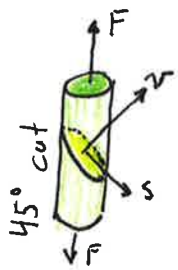
$$\boxed{|\sigma_s| = \underbrace{|T - (T\cdot\nu)\nu|}_{\text{Total Tracton vector minus the normal part}}}$$

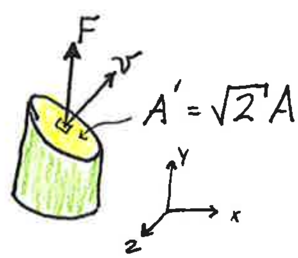
**The normal and shear stresses depend on the surface orientation**

Ex:




$$\boxed{\begin{aligned}\sigma_n &= \frac{F}{A} \\ \sigma_s &= 0\end{aligned}}$$




$$T(\nu) = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \frac{P}{\sqrt{2}} \frac{1}{A} \hat{\jmath} = \frac{1}{\sqrt{2}} \frac{P}{A} \hat{\jmath}$$

Compute $\nu = \frac{1}{\sqrt{2}} \hat{\imath} + \frac{1}{\sqrt{2}} \hat{\jmath}$

This is just stress transformation

- From above, $\sigma_n = T \cdot \nu = \left(0, \frac{1}{\sqrt{2}} \frac{P}{A}, 0\right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$$\sigma_n = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \frac{P}{A} = \boxed{\frac{1}{2} \frac{P}{A} = \sigma_n}$$

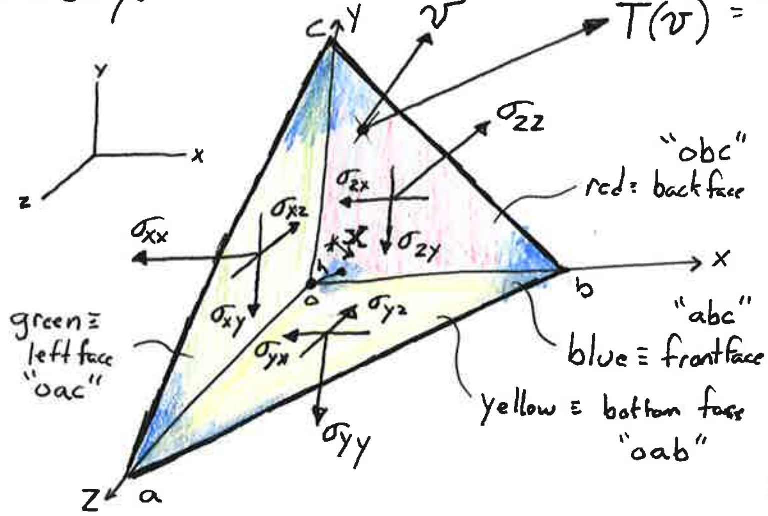
- Compute $\sigma_s = |T - (T\cdot\nu)\nu|$

$$\sigma_s = \left| \frac{1}{\sqrt{2}} \frac{P}{A} \hat{\jmath} - \left(\frac{1}{2}\frac{P}{A}\right)\left(\frac{1}{\sqrt{2}}\right)(\hat{\imath} + \hat{\jmath}) \right| = \left| \frac{1}{2\sqrt{2}} \frac{P}{A} \hat{\imath} + \frac{1}{2\sqrt{2}} \frac{P}{A} \hat{\jmath} \right|$$

$$\boxed{\sigma_s = \frac{1}{2} \frac{P}{A}}$$

# Cauchy's Formula

$$T(\nu) = T_x\hat{\imath} + T_y\hat{\jmath} + T_z\hat{k}$$




Volume of tetrahedra

$$V = \frac{1}{3}A \cdot h$$

Area of faces

$$A_{abc} = A$$
$$A_{oac} = A\nu_x$$
$$A_{oab} = A\nu_y$$
$$A_{obc} = A\nu_z$$

Summation of forces in x-direction

$$\Sigma F_x = 0 = \underbrace{T_x A}_{\text{front face}} - \underbrace{\sigma_{xx} A_{oac}}_{\text{left face}} - \underbrace{\sigma_{yx} A_{oab}}_{\text{bottom}} - \underbrace{\sigma_{zx} A_{obc}}_{\text{back}} + \underbrace{X V}_{\text{body force}} = 0 \quad \text{static}$$

$$= T_x A - \sigma_{xx} A\nu_x - \sigma_{yx} A\nu_y - \sigma_{zx} A\nu_z + X \tfrac{1}{3}Ah = 0$$

The area "A" is common, so apparently the size does not matter. Thus, as $h \to 0$

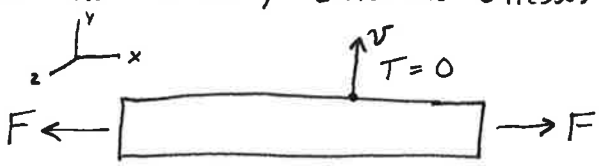
$$\boxed{T_x = \sigma_{xx}\nu_x + \sigma_{yx}\nu_y + \sigma_{zx}\nu_z}$$

Likewise,

$$\boxed{\begin{aligned} T_y &= \sigma_{xy}\nu_x + \sigma_{yy}\nu_y + \sigma_{zy}\nu_z \\ T_z &= \sigma_{xz}\nu_x + \sigma_{yz}\nu_y + \sigma_{zz}\nu_z \end{aligned}}$$

These Cauchy equations are required to be true on any surface of a body (internal or natural external).

Ex: Given a body in uniaxial loading and ~~no~~ traction at a point on the surface, determine stresses at that point.




$\nu = (0, 1, 0)$

$$T_x = 0 = \sigma_{xx}\nu_x + \sigma_{yz}\nu_y + \sigma_{zx}\nu_z = \sigma_{xx}\cdot 0 + \underbrace{\sigma_{yz}\cdot 1} + \sigma_{zx}\cdot 0$$

Thus, $\sigma_{yz}$ must be zero

$$T_y = 0 = \sigma_{xy}\cancel{\nu_x}^{0} + \underbrace{\sigma_{yy}\cancel{\nu_y}^{1}} + \sigma_{zy}\cancel{\nu_z}^{0}$$

$$\sigma_{yy} = 0$$

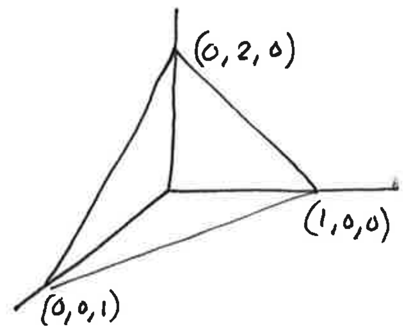
Notice this is on the surface.

$$T_z = 0 = \sigma_{xz}\cancel{\nu_x}^{0} + \underbrace{\sigma_{yz}\cancel{\nu_y}^{1}} + \sigma_{zz}\cancel{\nu_z}^{1}$$

$$\sigma_{yz} = 0$$

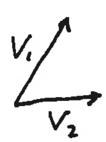
Ex: 2.1

$$\sigma = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & 0 \\ -4 & 0 & 5 \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \quad \text{and}$$




⓪ Find $\nu$

The cross product is a good way to generate normals. Pick 2 vectors on the surface.




$$V_1 = (0,2,0) - (0,0,1) = (0, 2, -1)$$
$$V_2 = (1,0,0) - (0,0,1) = (1, 0, -1)$$

$$\nu = \frac{V_2 \times V_1}{\underbrace{|V_2 \times V_1|}_{\text{magnitude}}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 2 & -1 \end{vmatrix} = \frac{2\hat{\imath} + 1\hat{\jmath} + 2\hat{k}}{\sqrt{2^2 + 1^2 + 2^2}} = \boxed{\frac{2}{3}\hat{\imath} + \frac{1}{3}\hat{\jmath} + \frac{2}{3}\hat{k}}$$

a) Find T components on plane described by $\nu$

$$T_x = \sigma_{xx}\nu_x + \sigma_{yx}\nu_y + \sigma_{zx}\nu_z = 1\cdot\frac{2}{3} + 0\cdot\frac{1}{3} + -4\cdot\frac{2}{3} = \frac{2}{3} - \frac{8}{3} = -\frac{6}{3} = -2$$

$$T_y = \sigma_{xy}\nu_x + \sigma_{yy}\nu_y + \sigma_{zy}\nu_z = 0\cdot\frac{2}{3} + 3\cdot\frac{1}{3} + 0\cdot\frac{2}{3} = 1$$

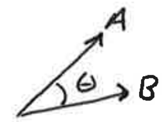
$$T_z = \sigma_{xz}\nu_x + \sigma_{yz}\nu_y + \sigma_{zz}\nu_z = -4\cdot\frac{2}{3} + 0\cdot\frac{1}{3} + 5\cdot\frac{2}{3} = -\frac{8}{3} + \frac{10}{3} = \frac{2}{3}$$

$$\boxed{T^{\nu} = \left(-2, 1, \tfrac{2}{3}\right)}$$

b) Find the magnitude of $T$

$$|Tv| = \sqrt{(-2)^2 + 1^2 + \left(\frac{2}{3}\right)^2} = \sqrt{4 + 1 + \frac{4}{9}} = \frac{7}{3}$$

c) Angle between $Tv$ and $v$



Use dot product: $\underbrace{A \cdot B = |A||B| \cos\theta}$

$$T(v) \cdot v = |Tv||v| \cos\theta$$

$$\underbrace{\left(-2, 1, \tfrac{2}{3}\right) \cdot \left(\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\right)} = \frac{7}{3} \cdot 1 \cdot \cos\theta$$

$$-2 \cdot \tfrac{2}{3} + \tfrac{1}{3} + \tfrac{4}{9} = -\frac{5}{9} = \frac{7}{3} \cos\theta \Rightarrow \cos\theta = \frac{-15}{63}$$

$$\theta = 76^\circ$$

d) $N = T \cdot v$

$$= \frac{-5}{9}$$

e) $S = |T - (N)v| = \left|\left(-2, 1, \tfrac{2}{3}\right) - \left(\tfrac{-5}{9}\right)\left(\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\right)\right|$

$$= \left|\left(-2 + \tfrac{10}{27}, 1 + \tfrac{5}{27}, \tfrac{2}{3} + \tfrac{2}{3}\right)\right|$$

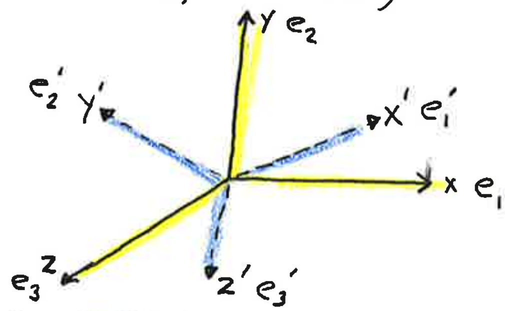
This gets messy

# Stress Transformation

Stress is a $2^{nd}$ order tensor, not a vector, you can not transform a stress by transforming each component. (i.e. $\sigma_{xx} \neq \sigma_{xx}\cos\theta + \sigma_{yy}\sin\theta$ etc)

Given a stress state.

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$




Convert/Transform from coordinates xyz to x'y'z'

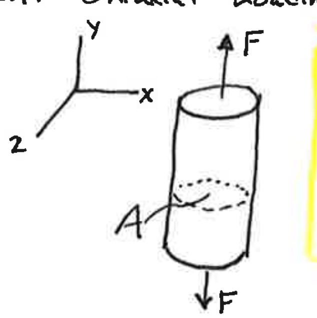
Define $A_{ij} \equiv e_i' \cdot e_j = \cos\left(\text{angle between } e_i' \text{ and } e_j\right)$     $e$ are unit vectors

The transformation is

$$\sigma' = A \sigma A^T$$

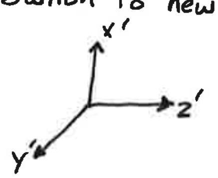
$$\begin{bmatrix} \sigma_{x'x'} & \sigma_{x'y'} & \sigma_{x'z'} \\ \sigma_{x'y'} & \sigma_{y'y'} & \sigma_{y'z'} \\ \sigma_{x'z'} & \sigma_{y'z'} & \sigma_{z'z'} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

## Ex: Uniaxial Loading




$$\boxed{\sigma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & F/A & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

Switch to new coordinate system




$A_{11} = e_x' \cdot e_x = 0$    $A_{23} = e_y' \cdot e_z = 1$

$A_{12} = e_x' \cdot e_y = 1$    $A_{31} = e_z' \cdot e_x = 1$

$A_{13} = e_x' \cdot e_z = 0$    $A_{32} = e_z' \cdot e_y = 0$

$A_{21} = e_y' \cdot e_x = 0$    $A_{33} = e_z' \cdot e_z = 0$

$A_{22} = e_y' \cdot e_y = 0$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Transformation

$$\sigma' = A\sigma A^T =$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & F/A & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & F/A & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} F/A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \sigma'}$$

Neat!