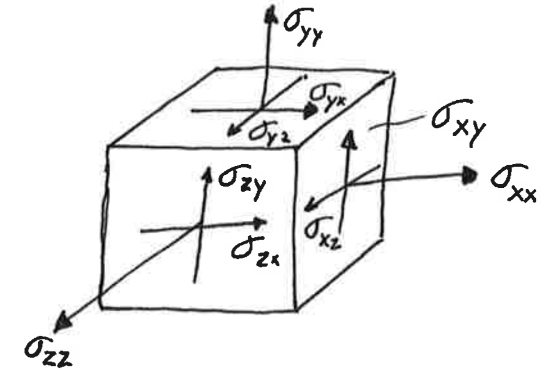
Three Dimensional Stress Analysis (principal and max shear stress)

Given a 3D stress state

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ & \sigma_{yy} & \sigma_{yz} \\ \text{symmetric} & & \sigma_{zz} \end{bmatrix}$$



The principal stresses are the eigenvalues of σ

The max shear is $\frac{1}{2}$ the difference between principal stresses

The principal plane directions are the eigenvectors of σ

Ex:

$$\sigma = \begin{bmatrix} 10 & -4 & 0 \\ -4 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Find principal stresses and max shear

① Eigenvalues of σ

$$|\sigma - I\lambda| = 0 \Rightarrow \begin{vmatrix} -\lambda+10 & -4 \\ -4 & -\lambda+6 \end{vmatrix} = \overbrace{\lambda^2}^{a=1} \overbrace{-\lambda 6 - \lambda 10}^{b=-16} + \overbrace{60 - 16}^{c=44}$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{16 \pm \sqrt{256 - 4 \cdot 44}}{2}$$

$$= 8 \pm 4.47$$

$\sigma_1 = 12.47$

$\sigma_2 = 3.53$

② Max Shear

$$\sigma_s = \frac{1}{2}(12.47 - 3.53) = 4.47$$

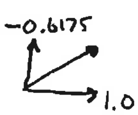
③ Direction (eigenvectors) $\quad \sigma v = \lambda v$

$$\begin{bmatrix} 10 & -4 \\ -4 & 6 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 12.47 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow 10v_1 - 4v_2 - 12.47 v_1 = 0$$

v can be scaled, so lets pick $v_1 = 1$

$$-2.47 = 4v_2 \Rightarrow v_2 = -0.6175$$

$$v_{\sigma_1} = \begin{pmatrix} 1 \\ -0.6175 \end{pmatrix}$$

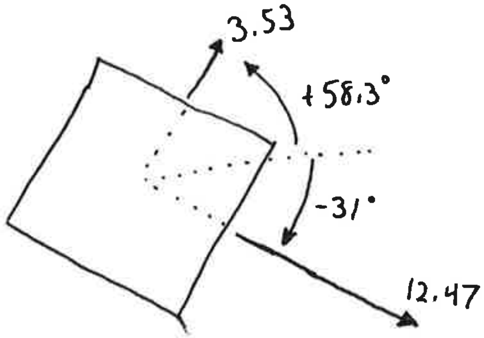


$$\phi = \operatorname{atan}\left(\frac{-0.6175}{1}\right) = -31.7^\circ$$

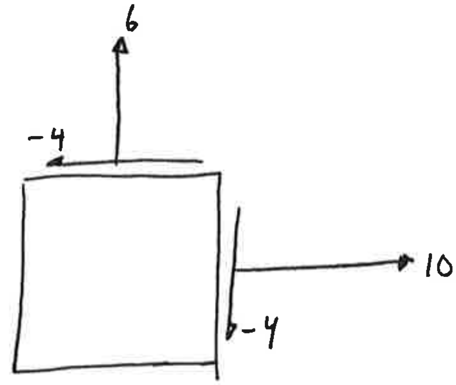
And $\quad 10v_1 - 4v_2 = 3.53 v_1 \Rightarrow 4v_2 = 6.47 \Rightarrow v_2 = 1.6175$

$$v_{\sigma_2} = \begin{pmatrix} 1 \\ 1.6175 \end{pmatrix} \qquad \phi = \operatorname{atan}(1.6175) = 58.3^\circ$$

(4) Draw Prin' stress



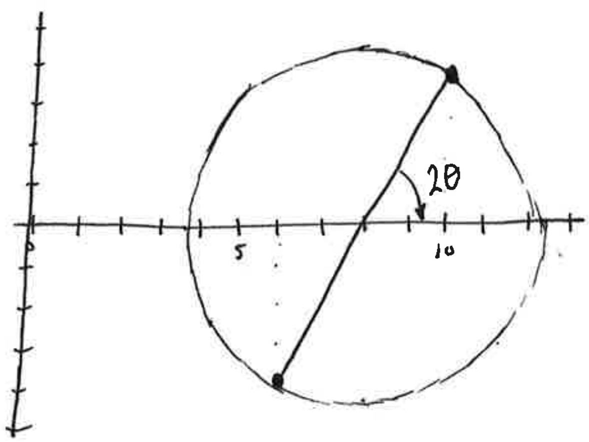
principal



original

(5) Mohr's Circle

Notice that MC is only applicable because σ is <u>planar</u> stress
MC <u>can not</u> transform arbitrary 3D stresses.



(1) Mark $(10,4)$
$(6,-4)$

(2) midpoint is $(8,0)$

(3) Radius is $\sqrt{(10-8)^2 + 4^2}$
$= 4.47$

(4) principal stresses

$8 + r = \boxed{12.47}$
$8 - r = \boxed{3.53}$

(5) Max Shear
$r = 4.47$

(6) principal angle

$-2\theta = \text{atan}\left(\frac{4}{10-8}\right) = \text{atan2}$

$-\theta = 31.7$

$\boxed{\theta = -31.7°}$

Same solution as more general 3D eigenvalue method

Alternative Method to calculate eigenvalues of 3x3 real sym matrix

By hand!

$$P_1 = \sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2$$

$$q = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

$$P_2 = (\sigma_{xx} - q)^2 + (\sigma_{yy} - q)^2 + (\sigma_{zz} - q)^2 + 2P_1$$

$$p = \sqrt{P_2/6}$$

$$[B] = [A - q[I]] = \begin{bmatrix} \sigma_{xx} - q & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - q & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - q \end{bmatrix}$$

$$r = \frac{\det(B)}{2p^3}$$

$$\phi = \frac{\operatorname{acos}(r)}{3}$$

$$\sigma_{p_1} = q + 2p\cos\phi$$

$$\sigma_{p_2} = q + 2p\cos\left(\phi + \frac{2}{3}\pi\right)$$

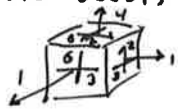
$$\sigma_{p_3} = q + 2p\cos\left(\phi + \frac{4}{3}\pi\right)$$

Calculator

eigval([σ])

- be careful, most calculators give eigenvectors that <u>might</u> not correspond to the eigenvalue order.

Ex: $\sigma = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix} \times 10^4 \text{ psi}$ If failure occurs at 80^{ksi}, is the stress state safe?



$$p_1 = 2^2 + 3^2 + 6^2 = 4 + 9 + 36 = 49$$

$$q = \frac{1}{3}(1+4+1) = 6/3 = 2$$

$$p_2 = (1-2)^2 + (4-2)^2 + (1-2)^2 + 2 \cdot 49 = 1 + 4 + 1 + 98 = 104$$

$$p = \sqrt{104/6} = 4.163$$

$$B = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 2 & 6 \\ 3 & 6 & -1 \end{bmatrix}$$ only subtract q on the diagonals!

$$\det B = -1(-2-36) - 2(-2-18) + 3(12-6) = 38 + 40 + 18 = 96$$

$$r = 96/2 \cdot \frac{1}{4.163^3} = 0.665$$ be sure use divide by p^3!

$$\phi = \frac{\operatorname{acos}(0.665)}{3} = 0.281$$ this is in radians

$$\sigma_{p_1} = 2 + 2 \cdot 4.163 \cdot \cos(0.281) = 10$$

$$\sigma_{p_2} = 2 + 2 \cdot 4.163 \cdot \cos(0.281 + \tfrac{2}{3}\pi) = -4$$

$$\sigma_{p_3} = 2 + 2 \cdot 4.163 \cdot \cos(0.281 + \tfrac{4}{3}\pi) = 0$$

principal stresses are 10, 4, 0

$10 > 8$

Failure of material
Not Safe

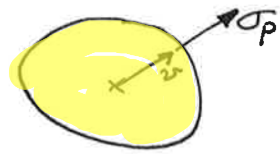
Use a calculator or computer when available.

You can use eigenvectors to find principal directions .... by hand is possible.

Why ~~is~~ are the eigenvalues the principal stresses?

① Definition of principal stress: Stress state where the shear is zero

σ_p on a principal plane of normal v



② Traction Vector

What is the traction vector associated with σ_p along v?

$$T(v) = \sigma_p v$$

$$T(v) = T_x v_x \hat{\imath} + T_y v_y \hat{\jmath} + T_z v_z \hat{k}$$
$$= \sigma_p v_x \hat{\imath} + \sigma_p v_y \hat{\jmath} + \sigma_p v_z \hat{k}$$

Notice, <u>no</u> shear so $T(v)$ is aligned with v.

③ Cauchy equations

x: $\underbrace{T_x(v)}_{\sigma_p v_x} = \sigma_{xx} v_x + \sigma_{xy} v_y + \sigma_{xz} v_z$ in $\hat{\imath}$ direction

Thus

$$\boxed{(\sigma_{xx} - \sigma_p) v_x + \sigma_{xy} v_y + \sigma_{xz} v_z = 0}$$

y: $\underbrace{T_y(v)}_{\sigma_p v_y} = \sigma_{xy} v_x + \sigma_{yy} v_y + \sigma_{yz} v_z$

Thus

$$\boxed{\sigma_{xy} v_x + (\sigma_{yy} - \sigma_p) v_y + \sigma_{yz} v_z = 0}$$

z: $\underbrace{T_z(v)}_{\sigma_p v_z} = \sigma_{xz} v_x + \sigma_{yz} v_y + \sigma_{zz} v_z$

Thus

$$\boxed{\sigma_{xz} v_x + \sigma_{yz} v_y + (\sigma_{zz} - \sigma_p) v_z = 0}$$

Put these 3 equations together and pull out v_x, v_y, v_z

$$\underbrace{\begin{bmatrix} \sigma_{xx}-\sigma_p & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\sigma_p & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\sigma_p \end{bmatrix}}_{\text{either this is zero}} \underbrace{\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}}_{\text{or this is zero}} = 0$$

④ Recall the definition of an eigenvalue

- Given matrix "A", solve $|A - \lambda I| = 0$ where λ is an eigenvalue and v is an eigenvector
- ~~[illegible]~~ $Av = \lambda v \Rightarrow A - \lambda I = 0$

⑤ Our green matrix above maps exactly to an eigenvalue problem where

$$A = \sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$

such that $A - \lambda I = [\sigma] - [\sigma_p] I$

$$= \begin{bmatrix} \sigma_{xx}-\sigma_p & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\sigma_p & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\sigma_p \end{bmatrix}$$

⑥ Eigenvalues of $[\sigma]$ are σ_p

Eigenvectors of $[\sigma]$ are principle directions of σ_p

Also $|[\sigma] - \sigma_p[I]| = \underbrace{-\sigma_p^3 + I_1\sigma_p^2 - I_2\sigma_p + I_3}_{\text{Which is where } I_1, I_2, I_3 \text{ come from as calculated in textbook}} = 0$

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{zz} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{xx} \end{vmatrix}$$

$$I_3 = \det(\sigma) = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}$$

Ex: Shear Stress (max) of Principal Directions $\sigma = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix}$ where $\sigma_p = (10, -4, 0)$

- $\sigma_p = 10$

$$([\sigma] - \sigma_p[I])v = 0$$

Need to find v such that this is true. Infinite # choices actually!

$$\begin{bmatrix} 1-10 & 2 & 3 \\ 2 & 4-10 & 6 \\ 3 & 6 & 1-10 \end{bmatrix}(v) = \underbrace{\begin{matrix} -9v_1 + 2v_2 + 3v_3 = 0 \\ 2v_1 - 6v_2 + 6v_3 = 0 \\ 3v_1 + 6v_2 - 9v_3 = 0 \end{matrix}}_{\text{This naturally has det = 0 (infinite \# solutions)}}$$

pick $v_{1_x} = 1$ and solve for v_{1_y} v_{1_z}

$$\begin{matrix} 2v_2 + 3v_3 = 9 \\ -6v_2 + 6v_3 = -2 \end{matrix} \Rightarrow \underbrace{\begin{bmatrix} 2 & 3 \\ -6 & 6 \end{bmatrix}\begin{pmatrix} v_{1_y} \\ v_{1_z} \end{pmatrix} = \begin{pmatrix} 9 \\ -2 \end{pmatrix}}_{\text{This \underline{can} be solved.}}$$

Inverse of 2x2 matrix $[M]$

- flip diagonal terms $(M'_{11} = M_{22})$
- $-$ off diagonals $(M'_{12} = -M_{12})$
- Divide by $\det(M)$

$$\begin{pmatrix} v_{1_y} \\ v_{1_z} \end{pmatrix} = \begin{bmatrix} 2 & 3 \\ -6 & 6 \end{bmatrix}^{-1}\begin{pmatrix} 9 \\ -2 \end{pmatrix} = \frac{\begin{bmatrix} 6 & -3 \\ 6 & 2 \end{bmatrix}}{12+18}\begin{pmatrix} 9 \\ -2 \end{pmatrix} = \frac{1}{30}\begin{pmatrix} 60 \\ 50 \end{pmatrix} = \begin{pmatrix} 2 \\ 5/3 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 5/3 \end{pmatrix}$$ Normalize by $1^2 + 2^2 + (5/3)^2 = \sqrt{7.777} = 2.788$

$$\boxed{v_1 = \begin{pmatrix} 0.358 \\ 0.717 \\ 0.597 \end{pmatrix} \text{ for } \sigma_p = 10}$$

- $\sigma_p = -0$ $\quad v_2 = \begin{pmatrix} -0.894 \\ 0.447 \\ 0 \end{pmatrix}$

- $\sigma_p = -4$ $\quad v_3 = \begin{pmatrix} -0.267 \\ -0.535 \\ 0.802 \end{pmatrix}$

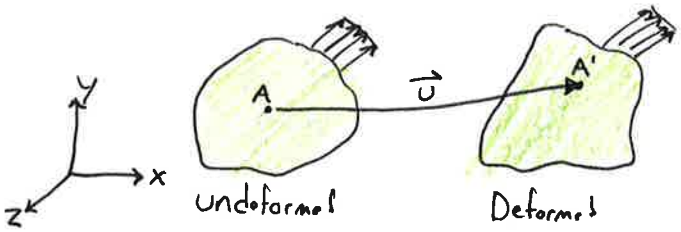
- Shear Stress

$$\sigma_{s_1} = \left|\frac{10-4}{2}\right| = 3$$

$$\sigma_{s_2} = \left|\frac{4-0}{2}\right| = 2$$

$$\sigma_{s_3} = \left|\frac{10-0}{2}\right| = 5$$

Deformation Analysis



$$\vec{U} = U\hat{i} + V\hat{j} + W\hat{k}$$

Vector scalar

Given the body is a continuum, $\vec{U} = \vec{U}(x,y,z)$

We can take spatial derivatives

$$\begin{matrix} \frac{du}{dx}, & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz} \\ \frac{dw}{dx} & \frac{dw}{dy} & \frac{dw}{dz} \end{matrix}$$

Write the deformation of A to A'

$$A' = A + \vec{U}$$

If the body is a continuum, then in the nearby neighborhood of A

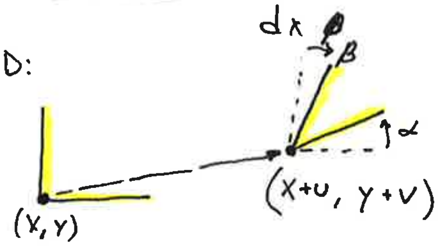
$$x \longrightarrow x + u + \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz$$

Engineering Strain

$$\varepsilon_{xx} \approx \frac{\left(\frac{du}{dx}dx + u + x\right) - (u+x)}{dx} = \frac{du}{dx}$$

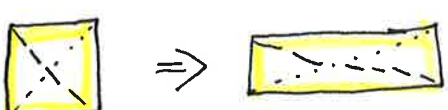
$$\boxed{\varepsilon_{xx} \equiv \frac{du}{dx}}$$

In 2D:

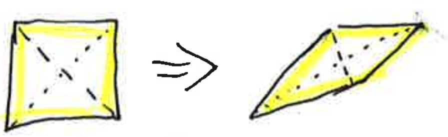


$$\varepsilon_{xy} = \alpha + \beta = \frac{dv}{dx} + \frac{du}{dy}$$

Visually



$\varepsilon_{xx} > 0$



$\varepsilon_{xy} > 0$