Differential Equations of Equilibrium

Lesson 6

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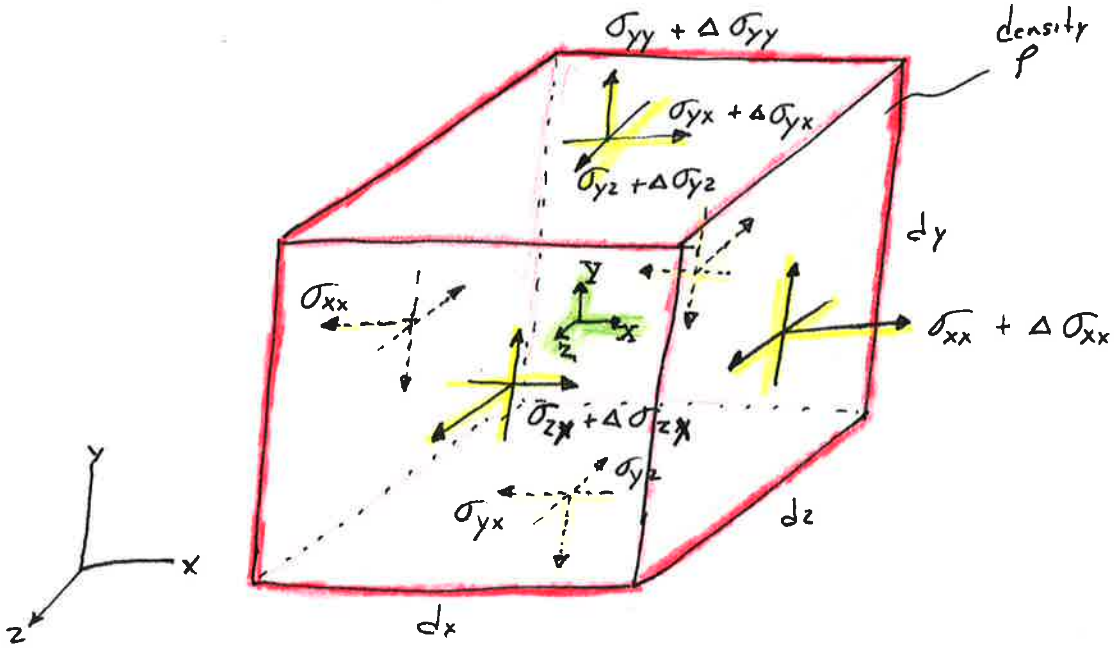
Small differential body of volume $\Delta x \; \Delta y \; \Delta z$



In the x direction, $\Sigma F_x = m a_x = (\rho\, dx\, dy\, dz)\, \cancel{a_x}$ 0 if static

$$\begin{aligned}\Sigma F_x = X \cdot V &+ (\sigma_{xx} + \Delta\sigma_{xx})\,dz\,dy - (\sigma_{xx})\,dz\,dy \\ &+ (\sigma_{yx} + \Delta\sigma_{yx})\,dx\,dz - (\sigma_{yx})\,dx\,dz \\ &+ \underbrace{(\sigma_{zx} + \Delta\sigma_{zx})\,dx\,dy}_{+\text{ faces}} \; \underbrace{- (\sigma_{zx})\,dx\,dy}_{-\text{ faces}}\end{aligned} = \begin{aligned}&\Delta\sigma_{xx}\,dz\,dy \\ &+ \Delta\sigma_{yx}\,dx\,dz \\ &+ \Delta\sigma_{zx}\,dx\,dy \\ &+ X\,dx\,dy\,dz\end{aligned}$$

$$= \frac{\Delta\sigma_{xx}}{\Delta x}\Delta x\, dy\, dz + \frac{\Delta\sigma_{yx}}{\Delta y}dx\,\Delta y\,dz + \frac{\Delta\sigma_{zx}}{\Delta z}dx\,dy\,\Delta z + X\,dx\,dy\,dz$$

Since $dx\,dy\,dz$ is a like term and the volume is not zero

$$\frac{\Delta\sigma_{xx}}{\Delta x} + \frac{\Delta\sigma_{yx}}{\Delta y} + \frac{\Delta\sigma_{zx}}{\Delta z} + X = 0$$

As the volume decreases in the limit $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$

$$\boxed{\frac{d\sigma_{xx}}{dx} + \frac{d\sigma_{xy}}{dy} + \frac{d\sigma_{xz}}{dz} + X = 0}$$

Likewise, the y and z directions are

$$\boxed{\frac{d\sigma_{xy}}{dx} + \frac{d\sigma_{yy}}{dy} + \frac{d\sigma_{zy}}{dz} + Y = 0}$$

$$\boxed{\frac{d\sigma_{xz}}{dx} + \frac{d\sigma_{yz}}{dy} + \frac{d\sigma_{zz}}{dz} + Z = 0}$$

Moments

X moment

$$\Sigma M_x = 0 = \underbrace{(\sigma_{yz} + \Delta\sigma_{yz})}_{\text{stress}} \underbrace{dx\,dz}_{\text{area}} \underbrace{\frac{dy}{2}}_{\text{arm}} - (\sigma_{zy} + \Delta\sigma_{zy})\,dx\,dy\,\frac{dz}{2}$$

$$+ (\sigma_{yz})\,dx\,dz\,\frac{dy}{2} - (\sigma_{zy})\,dx\,dy\,\frac{dz}{2}$$

Divide by volume $\left(\frac{dx\,dy\,dz}{2}\right)$ and collect like terms

$$\sigma_{yz} + \Delta\sigma_{yz} - \sigma_{zy} - \Delta\sigma_{zy} + \sigma_{yz} - \sigma_{zy} = 0$$

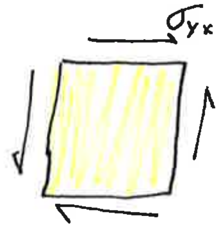
so that

$$\underbrace{2\sigma_{yz} - 2\sigma_{zy}}_{} + \underbrace{\Delta\sigma_{yz} - \Delta\sigma_{zy}}_{\rightarrow 0 \text{ in limit as } \Delta x \Delta y \Delta z \rightarrow 0} = 0$$

$$\boxed{\sigma_{yz} = \sigma_{zy}}$$

likewise

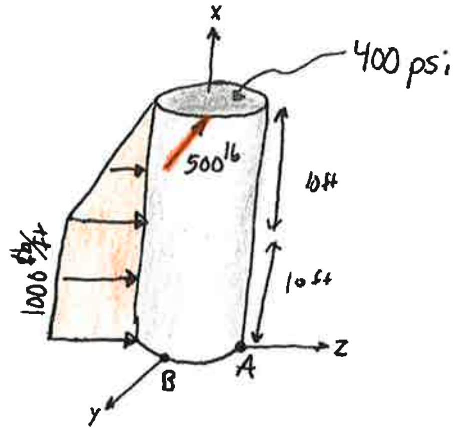
$$\boxed{\begin{aligned} \sigma_{xy} &= \sigma_{yx} \\ \sigma_{zx} &= \sigma_{xz} \end{aligned}}$$

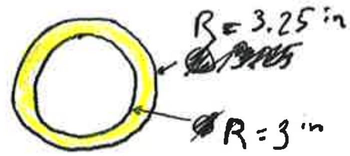


$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

9 stresses
+
6 equations!

Ex 1.12





① 2nd moment of inertia (Table A.2 p499)

- Exact

Circle of $R = 3.25$ — Circle of $R = 3.0$ in

$$I_{yy} = \frac{\pi R^4}{4} \Rightarrow I_{yy} = I_{zz} = \frac{\pi}{4}\left(R_o^4 - R_I^4\right) = \frac{\pi}{4}\left(3.25^4 - 3^4\right)$$

$$= 24.00 \text{ in}^4$$

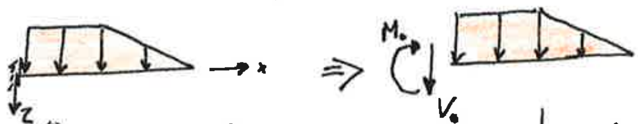
- Approximate $(t \ll r)$

$$I_{yy} = \pi r^3 t = \pi (3.125)^3 \cdot 0.25 = 23.96 \text{ in}^4$$

be careful when t ≮ r and be careful of radius

② Moment and shear at $x=0$ at "A"

- Integral



$$M_o + \underbrace{\int_0^{10} 1000\, x\, dx}_{50000} + \underbrace{\int_{10}^{20} (2000 - 100x)\, x\, dx}_{\frac{200000}{3}}$$

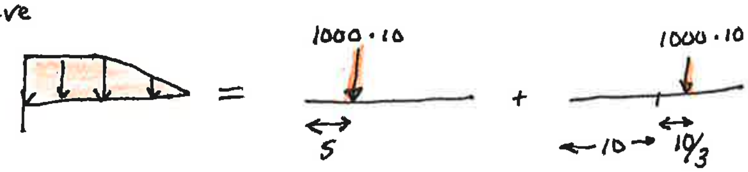
$$M_{y_0} = -116666 \text{ ft lbf}$$

$$\Sigma F_z = 0 = V_0 + \int_0^{20} F'\, dx$$

$$V_{x_0} + \underbrace{\int_0^{10} 1000\, dx}_{1000 \cdot 10} + \underbrace{\int_{10}^{20} (2000 - 100x)\, dx}_{2000(10) - \frac{100}{2}(20^2 - 10^2)} = 0$$

$$V_{20} = -15000 \text{ lbf}$$

- Intuitive



$$= 5 \cdot 1000 \cdot 10 + \frac{1000 \cdot 10}{2}\left(10 + \frac{10}{3}\right)$$

$$= -116666 \text{ ft lbf}$$

Easier!

③ Stresses at A due to bending moment M_{y_0}

$$\sigma_x = +\frac{Mz}{I_{yy}} = -\frac{M_{y_0} \cdot \overset{3.25}{z}}{I_{yy}} = -\frac{116666 \text{ ft lbf} \mid 3.25 \text{ in} \mid 12 \text{ in}}{\mid 24 \text{ in}^4 \mid \text{ft}}$$

$$= -15.8 \text{ ksi}$$

④ Shear at A due to 1000 lbf/ft loading

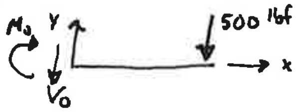
$$Q = \int y\,dA = 0 \quad \text{since on bottom (z) edge} \quad \sigma_{xy} = 0$$

⑤ pressure loading at A due to 400 psi pressure

$$\sigma_{hoop} = \frac{pr}{t} = \frac{400\ \text{lbf}}{\text{in}^2}\cdot\frac{3.125\ \text{in}}{0.25\ \text{in}} = 5\ \text{ksi}$$

$$\sigma_L = \frac{pr}{2t} = 2.5\ \text{ksi}$$

⑥ Moment and shear due to 500 lbf "y" load



$$V_0 = -500\ \text{lbf} \qquad M_0 = -10000\ \text{lbf ft}$$

⑦ Stresses at B due to "y" load

$$\sigma_x = +\frac{M_z\,y}{I_{zz}} = \frac{10000\ \text{lbf ft}\cdot 3.25\ \text{in}}{24\ \text{in}^4}\cdot\frac{12\ \text{in}}{\text{ft}} = -16.25\ \text{ksi}$$

⑧ Shear at A due to y load

$$\sigma_{shear} = \frac{VQ}{It} \qquad Q = \int z\,dA$$

$$= \int_3^{3.25}\int_0^{\pi} r^2 \sin\theta\, d\theta\, dr \qquad A = r\,d\theta\,dr \quad Z = r\sin\theta$$

$$= \frac{r^3}{3}\cdot 2 = 4.885\ \text{in}^3$$

$$= \frac{-500\ \text{lbf}\cdot 4.885\ \text{in}^3}{24\ \text{in}^4\cdot 0.5\ \text{in}}$$

$$= 203\ \text{psi} \quad \underline{\text{tiny}}$$

⑨ Shear at B due to "z" loading

$$\sigma_{shear} = \frac{-15000\ \text{lbf}\cdot 4.885\ \text{in}^3}{24\ \text{in}^4\cdot 0.5\ \text{in}} = -6.1\ \text{ksi}$$

⑩ at A:

$$\sigma_{xx} = \sigma_L + \sigma_x = 2.5 - 15.8 = -13.3$$

$$\sigma_{yy} = \sigma_{hoop} = 5\ \text{ksi}$$

$$\sigma_{zz} = p = 0.4\ \text{ksi}$$

$$\sigma_{xy} = 0.2\ \text{ksi}$$

$$\sigma = \begin{bmatrix} -13.3 & 0.2 & 0 \\ 0.2 & 5 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}$$

at B:

$$\sigma_{xx} = \sigma_x + \sigma_L = 2.5 - 16.25 = -13.75$$

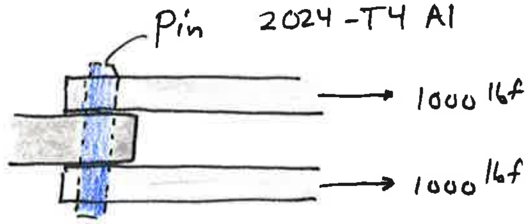
$$\sigma_{yy} = p = 0.4\ \text{ksi}$$

$$\sigma_{zz} = \sigma_{hoop} = 5\ \text{ksi}$$

$$\sigma_{xz} = -6.1\ \text{ksi}$$

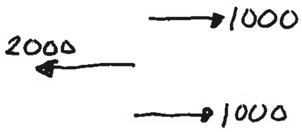
$$\sigma = \begin{bmatrix} -13.75 & 0 & -6.1 \\ 0 & 0.4 & 0 \\ -6.1 & 0 & 5 \end{bmatrix}$$

Ex. 1.13

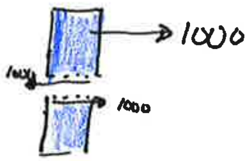


Safety factor = 2.0

① FBD of connector



② At interface, the shear is 1000 lbf



③ Area

$$A = \frac{\pi D^2}{4}$$

④ 2024-T4 properties

Table A.1

Yield tensile ≈ 40 ksi

$\underbrace{\text{Shear yield} \approx 55\%\ \text{Tensile Yield}}_{\text{Not in book.}} \approx 22\ \text{ksi}$

⑤ S.F and Shear stress

$$\tau = \frac{F}{A} = \frac{4F}{\pi D^2} \quad \text{and} \quad \tau_{SF} = \tau \cdot SF = \frac{4F}{\pi D^2} \cdot SF$$

and

$$\tau_{yield} = 22\,\text{ksi} = \frac{4F}{\pi D^2} \cdot SF$$

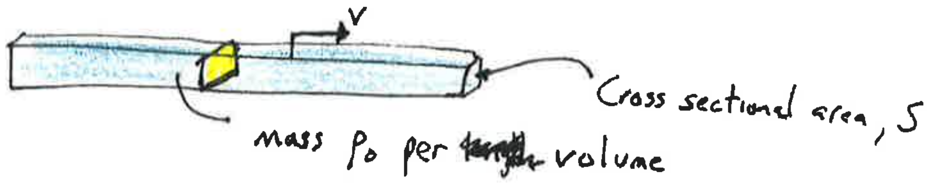
Solve for D

$$D = \sqrt{\frac{4F}{\pi} \frac{SF}{22\,\text{ksi}}} = \sqrt{\frac{4 \cdot 1000\,\text{lbf} \cdot 2.0}{\pi \cdot 22\,\text{ksi}} \cdot \frac{\text{ksi}\ \text{in}^2}{1000\,\text{lbf}}}$$

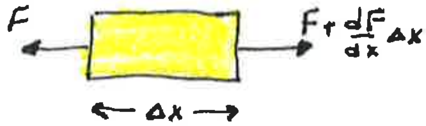
$$\boxed{D = 0.34\ \text{in}}$$

Use a 3/8 in bolt?

Longitudinal Waves in a rod



- Δx element Forces



- Newton's law

$$\Sigma F = ma = S \rho_0 \Delta x \frac{d^2 v}{dt^2}$$

- Stress - strain relationship

$$\sigma = \frac{F}{\underset{=S}{A}} = E \frac{dv}{dx} \quad \Rightarrow \quad F = SE \frac{dv}{dx}$$

- Combined

$$S \rho_0 \Delta x \frac{d^2 v}{dt^2} = -F + F + \frac{dF}{dx}\Delta x$$

$$= \frac{d}{dx}\left(S E \frac{dv}{dx}\right)\Delta x$$

- Cancel Δx and S (if cross section is constant), divide by ρ_0

$$\boxed{\frac{d^2 v}{dt^2} = \frac{E}{\rho_0} \frac{d^2 v}{dx^2}}$$ A wave equation $$\boxed{U_{tt} = c\, U_{xx}}$$

- Wave velocity is $\sqrt{E/\rho_0}$

- Steel bar.

$$E \approx 30 \times 10^6 \text{ psi} \qquad \rho_0 \approx 0.28 \frac{\text{lb}}{\text{in}^3} \approx 7 \times 10^{-4} \frac{\text{sl inch}}{\text{in}^3}$$

$$c \approx \sqrt{\frac{30 \times 10^6 \text{ psi}}{} \cdot \frac{\text{in}^3}{0.28 \text{ lbm}} \cdot \frac{32.2}{\cancel{\text{ }}} \cdot \frac{12}{}} \approx 17000 \text{ ft/s}$$

$$\approx 5000 \text{ m/s}$$

FAST

We can measure E with this!

D'Alembert Solution to W.E

$$U_{tt} = c^2 U_{xx} \qquad -\infty < x < \infty \qquad 0 < t < \infty$$

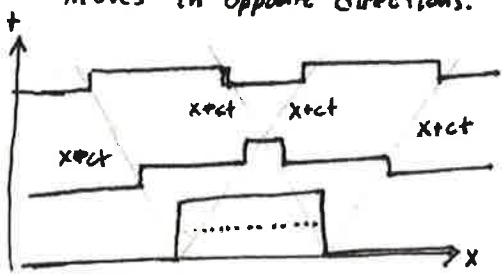
$$U(x,0) = f(x)$$

$$U_t(x,0) = g(x)$$

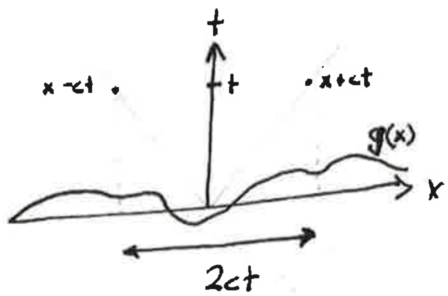
The general solution is

$$U(x,t) = \underbrace{\frac{1}{2}\Big(f(x-ct) + f(x+ct)\Big)}_{\text{Split IC displacement into 2 equal halves. Each part moves in opposite directions.}} + \underbrace{\frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi}_{\text{Average of IC velocity @ t within a domain of influence.}}$$

Displacement

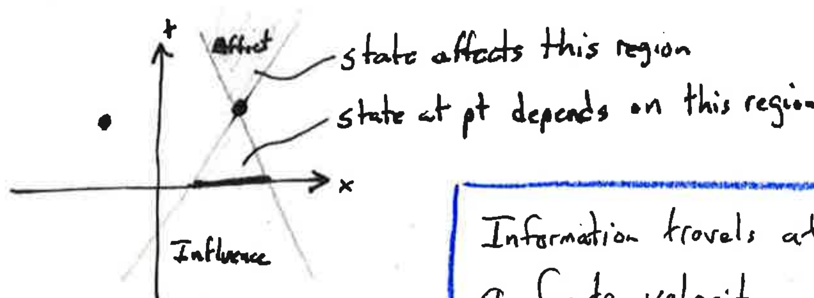


Velocity



Zone of Action

Wave equations have a domain of influence and effect based on a finite wave speed.



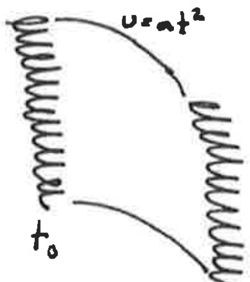
Information travels at a finite velocity

Drop a Slinky.  

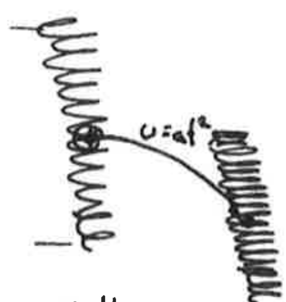
tiny.cc/GES 554-Slinky Slow
tiny.cc/GES 554-Slinky Intro

Wave equation
$$\frac{d^2 v}{dt^2} = \frac{E}{\rho_0}\frac{\partial^2 v}{\partial x^2}$$

Hold a slinky by one end. Let the other end stretch downward.
Let go.... what happens?

a)


no collapse, accelerate down uniformly at g

b)


collapse, center accelerates downward at g

c)

collapse, no acceleration downward.

d) Something else. ✓

Comments:

- Rotational wave speed is faster than elastic wave speed.
- Everything behaves this way.
- Center of mass still behaves as a rigid body. CM is not at center of length!