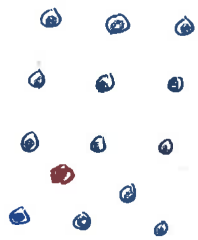
Isotropic Materials

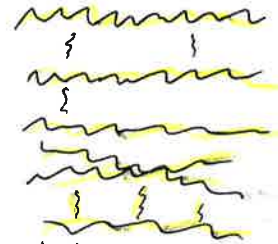
Material properties are independent of coordinate direction for isotropic materials

atomic/microscopic view

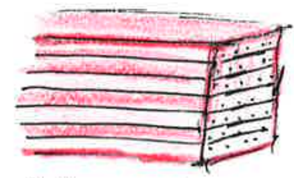


Metals ✓

Some polymers ✓



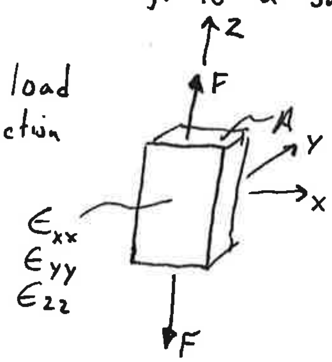
But not other polymers ✗



Definitely not unidirectional composites or fiber organics (wood) ✗

Apply a set of loadings to a sample of the material

① Uniaxial load Z direction



$\sigma_{zz} = \frac{F}{A}$ results in a $\underbrace{\text{strain } \epsilon_{zz}}_{\text{~~transverse~~ longitudinal}}$ and $\underbrace{\epsilon_{yy} \text{ and } \epsilon_{xx}}_{\text{transverse}}$

We know and see that the longitudinal and transverse strains in uniaxial loading are related with

$$\boxed{\text{Poisson's Ratio} \quad \nu_{ab} = -\frac{\epsilon_b}{\epsilon_a}}$$

a: direction of stress
b: transverse direction

For this Hookean (linear) range, the ratio of stress to strain is

$$\boxed{\text{Young's modulus} \equiv E_a = \frac{\sigma_a}{\epsilon_a}}$$

For our previous vector strain/stress equation

$$(\sigma) = [D](\epsilon) \quad \text{and} \quad (\epsilon) = [C](\sigma)$$

This load gives

$$\epsilon_{zz} = \frac{\sigma_{zz}}{E}, \quad \epsilon_{xx} = -\frac{\nu}{E}, \quad \epsilon_{yy} = -\frac{\nu}{E}$$

No shear $\epsilon_{xy} = 0 = \epsilon_{yz} = \epsilon_{xz}$

b/c isotropic

② Uniaxial load in x direction

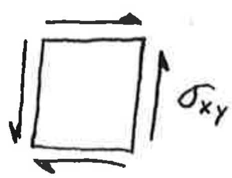
Isotropic material, so almost identical to ① except indicies are switched

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E}, \quad \varepsilon_{yy} = -\frac{\nu}{E}, \quad \varepsilon_{zz} = \frac{-\nu}{E}$$

③ Uniaxial load in y

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E}, \quad \varepsilon_{xx} = -\frac{\nu}{E}\sigma_{yy}, \quad \varepsilon_{zz} = -\frac{\nu}{E}\sigma_{yy}$$

④ Shear loading



σ_{xy} results in a strain ε_{xy}. The ratio is

$$\boxed{\text{shear modulus} \equiv G \equiv \frac{\sigma_{xy}}{\varepsilon_{xy}}}$$

In general,

$$\varepsilon_{xy} = \frac{1}{G}\sigma_{xy}, \quad \varepsilon_{xz} = \frac{1}{G}\sigma_{xz}, \quad \varepsilon_{yz} = \frac{1}{G}\sigma_{yz}$$

The strains are in the linear region. Add strains from each test

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{\sigma_{xx}}{E} - \frac{\nu}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz} \\ \varepsilon_{yy} &= \frac{\sigma_{yy}}{E} - \frac{\nu}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{zz} \\ \varepsilon_{zz} &= \frac{\sigma_{zz}}{E} - \frac{\nu}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} \end{aligned}\right\}$$

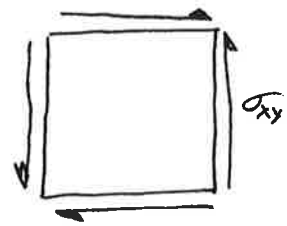
$$\left.\begin{aligned} \varepsilon_{xy} &= \frac{1}{G}\sigma_{xy} \\ \varepsilon_{xz} &= \frac{1}{G}\sigma_{xz} \\ \varepsilon_{yz} &= \frac{1}{G}\sigma_{yz} \end{aligned}\right\}$$

$$\underbrace{\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{pmatrix}}_{\text{strain vector}} = \underbrace{\begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & & & \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & & & \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & & & \\ & & & \frac{1}{G} & & \\ & & & & \frac{1}{G} & \\ & & & & & \frac{1}{G} \end{bmatrix}}_{\text{compliance matrix}} \underbrace{\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix}}_{\text{stress vector}}$$

Poisson's ratio, Young's modulus, and the shear modulus are not independent.

<u>Any 2 describes the 3rd.</u>

① Pick a shear loading



$\sigma_{xy} \neq 0$ and $\sigma_{ij} = 0$ otherwise

The stress state is $\sigma = \begin{bmatrix} 0 & \sigma_{xy} & 0 \\ \sigma_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The strain state is $\varepsilon_{xy} = \frac{1}{G}\sigma_{xy}$

② Compute the principal stresses

2x2 matrix, so I'll do it by hand. $|I\lambda - A| = 0 = \begin{vmatrix} \lambda & -\sigma_{xy} \\ -\sigma_{xy} & \lambda \end{vmatrix} = \lambda^2 - \sigma_{xy}^2$

$\lambda = \pm\, \sigma_{xy}$

$\sigma_{p_1} = +\sigma_{xy}$, $\sigma_{p_2} = -\sigma_{xy}$

③ Principal Directions

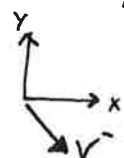
$+ \quad [A]v = \lambda v \Rightarrow \begin{bmatrix} 0 & \sigma_{xy} \\ \sigma_{xy} & 0 \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sigma_{xy}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow \sigma_{xy} V_2 = \sigma_{xy} V_1$

$V_2 = V_1$

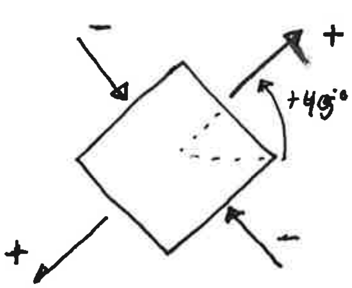
$V^+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$- \quad (A)v = \lambda v \Rightarrow \sigma_{xy} V_2 = -\sigma_{xy} V_1 \Rightarrow V_2 = -V_1$

$V^- = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



④ This is identical to the stress loading of ~~[scribbled out]~~ positive σ_{xx} in the $+45^\circ$ and negative σ_{yy} in the -45°



Transforming the strain from xy to the new $\pm 45^\circ$ axis gives (eq. 2.51a)

$$\varepsilon_{x'x'} = \cancel{\varepsilon_{xx}}^{0} \cos^2\theta + \cancel{\varepsilon_{yy}}^{0} \sin^2\theta + \varepsilon_{xy} \sin\cancel{\theta}^{45^\circ} \cos\cancel{\theta}^{45^\circ}$$

$$= \varepsilon_{xy} \tfrac{1}{2}$$

And, $\varepsilon_{x'x'}$ is $\underbrace{\text{(from above)}}$ $= \frac{1}{E}\underbrace{\sigma_{x'x'}}_{\sigma_{xy}} - \frac{\nu}{E}\underbrace{\sigma_{y'y'}}_{-\sigma_{xy}} = \varepsilon_{xy}\frac{1}{2} = \frac{1}{2}\frac{1}{G}\sigma_{xy}$

So, $\frac{1}{E} + \frac{\nu}{E} = \frac{1}{2}\frac{1}{G}$

Solve for G

$$\boxed{G = \frac{1}{2}\frac{E}{1+\nu}}$$

The 3D isotropic Hookean strain – stress equation is

$$\boxed{\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{pmatrix} = \frac{1}{E} \underbrace{\begin{bmatrix} 1 & -\nu & -\nu & & & \\ -\nu & 1 & -\nu & & & \\ -\nu & -\nu & 1 & & & \\ & & & 2(1+\nu) & & \\ & & & & 2(1+\nu) & \\ & & & & & 2(1+\nu) \end{bmatrix}}_{\text{Notice that the stresses and strains are block diagonal for normal and shear}} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix}}$$

Eq. 3.38 in Allen + Haisler

$\Rightarrow$ "Normal stresses contribute to only normal strains"
"Same for shear"

The inverse of the compliance matrix is the modulus matrix
[C] [D]

In general, the inverse of a 3×3 symmetric matrix is known

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} \qquad \det(M) = m_{11} \cdot f_{11} + m_{21} \cdot f_{21} + m_{13} \cdot f_{13}$$

$$F = M^{-1} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{bmatrix}$$

$$f_{11} = m_{33} \cdot m_{22} - m_{23}^2 \qquad f_{22} = m_{33} \cdot m_{11} - m_{13}^2$$

$$f_{12} = m_{13} \cdot m_{23} - m_{33} \cdot m_{12} \qquad f_{23} = m_{12} \cdot m_{13} - m_{11} \cdot m_{23}$$

$$f_{13} = m_{12} \cdot m_{23} - m_{13} \cdot m_{22} \qquad f_{33} = m_{11} \cdot m_{22} - m_{12}^2$$

Apply to [C] to give

$$\boxed{\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & & \\ \nu & \nu & 1-\nu & & & \\ & & & \frac{1-2\nu}{2} & & \\ & & & & \frac{1-2\nu}{2} & \\ & & & & & \frac{1-2\nu}{2} \end{bmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{pmatrix}}$$

Eq. 3.39 (put a tab in your book)

Ex:

Using Strain gauges on an aircraft's wing surface, you determine

$\varepsilon_{xx} = 2000$ micro strains

$\varepsilon_{yy} = 5700$ "

$\varepsilon_{xy} = 3000$ "

Not to scale

Determine if the stress state is safe for 6061-T6 Al sheet as the skin.

① Strains

$\varepsilon_{xx} = 2000$ microstrain $= 2000 \cdot 10^{-6} = 0.002$ ← sometimes 0.2%

$\varepsilon_{yy} = 5700$ " $= 0.0057$

$\varepsilon_{xy} = 0.003$

② Isotropic Hookean stress/strain

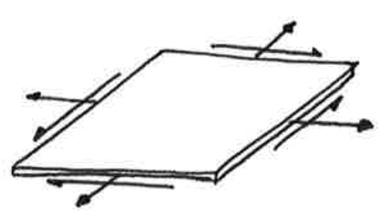
$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & & \\ \nu & \nu & 1-\nu & & & \\ & & & \frac{1-2\nu}{2} & & \\ & & & & \frac{1-2\nu}{2} & \\ & & & & & \frac{1-2\nu}{2} \end{bmatrix} \begin{pmatrix} 0.002 \\ 0.0057 \\ 0 \\ 0 \\ 0 \\ 0.003 \end{pmatrix}$$

③ Material

Al $\quad E = 9.9 \times 10^6$ psi $\quad \nu = 0.33 \quad Y = 36$ ksi

STOP! What is wrong here?

We made a fundamental error....



What is the loading?

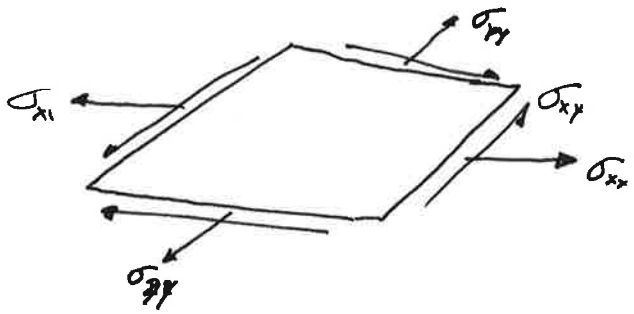
Surface Traction?

Plane stress....

Above, we said $\varepsilon_{zz} = 0$, but rather $\sigma_{zz} = 0$!

2D "planar" stress loading

(i.e. No traction in z direction)
free to expand in z direction



No traction in z means $\sigma_{zz} = 0$. Take initial Hookean strain-stress equation

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{pmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & \cancel{-\nu} & & & \\ -\nu & 1 & -\nu & & & \\ -\nu & -\nu & 1 & & & \\ & & & \cancel{2(1+\nu)} & & \\ & & & & \cancel{2(1+\nu)} & \\ & & & & & 2(1+\nu) \end{bmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \cancel{\sigma_{zz}} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix} \leftarrow \text{zero}$$

Now $\boxed{\varepsilon_{zz} = \frac{1}{E}\left(-\nu\sigma_{xx} - \nu\sigma_{yy}\right)}$ and we can cross it out of the above

Now a 3×3

$$\boxed{\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\nu & \\ -\nu & 1 & \\ & & 2(1+\nu) \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}}$$

Invert this to give: (using upper 2x2 block diagonal)

$$\boxed{\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & \\ \nu & 1 & \\ & & \frac{1-\nu}{2} \end{bmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix}}$$

$$M_{2\times 2}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc}$$

so

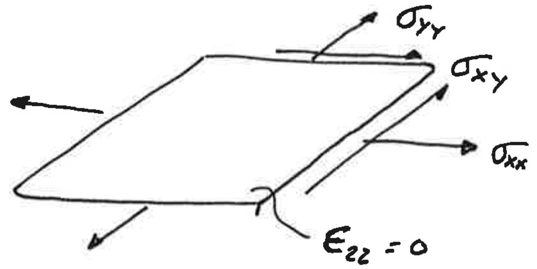
$$\left(\frac{1}{E}\begin{bmatrix} 1 & -\nu \\ -\nu & 1 \end{bmatrix}\right)^{-1} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}$$

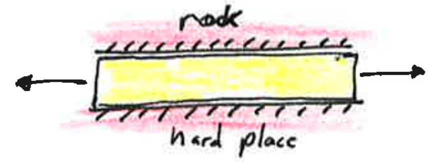
Notice and Warning

You can not just cross off terms (rows and columns) if the constraint is on the left hand side of the equation

2D planar strain loading (i.e. constrained z-direction)

unknown σ_{zz}





Just use the 3D version and cross off ε_{zz} terms (from the RHS)

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \cancel{\sigma_{yz}} \\ \cancel{\sigma_{xz}} \\ \sigma_{xy} \end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \cancel{\nu} & & & \\ \nu & 1-\nu & \cancel{\nu} & & & \\ \nu & \nu & \cancel{1-\nu} & & & \\ & & & \cancel{\frac{1-2\nu}{2}} & & \\ & & & & \cancel{\frac{1-2\nu}{2}} & \\ & & & & & \frac{1-2\nu}{2} \end{bmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \cancel{\varepsilon_{zz}} \\ \cancel{\varepsilon_{yz}} \\ \cancel{\varepsilon_{xz}} \\ \varepsilon_{xy} \end{pmatrix}$$

pull out the σ_{zz} row

$$\boxed{\sigma_{zz} = \frac{\nu E}{(1+\nu)(1-2\nu)} \left(\varepsilon_{xx} + \varepsilon_{yy} \right)}$$

Write 3×3

$$\boxed{\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \\ \nu & 1-\nu & \\ & & \frac{1-2\nu}{2} \end{bmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix}}$$

Invert to give ε strains

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} = \frac{(1+\nu)(1-2\nu)}{E} \frac{\begin{bmatrix} 1-\nu & -\nu & \\ -\nu & 1-\nu & \\ & & \frac{2}{1-2\nu} \end{bmatrix}}{1-2\nu} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}$$

$$\boxed{= \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & \\ -\nu & 1-\nu & \\ & & \frac{2}{1-2\nu} \end{bmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}}$$

Notice that this is not the same as the book.
Typo in 3.43 σ_{xy} term

Retry our example

We now recognize that since we have 2D plane stress, we must use the appropriate form of stress - strain relation.

① 2D plane stress

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & \\ \nu & 1 & \\ & & \frac{1-\nu}{2} \end{bmatrix}\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix}$$

② Material

$E = 9.9\times10^6\,psi$, $\nu = 0.33$, $Y = 36^{ksi}$

$1-\nu^2 = 0.891$, $\frac{1-\nu}{2} = 0.335$

③ plug #s

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \frac{9.9\times10^6\,psi}{0.891}\begin{bmatrix} 1 & 0.33 & \\ 0.33 & 1 & \\ & & 0.335 \end{bmatrix}\begin{pmatrix} 0.002 \\ 0.0057 \\ 0.003 \end{pmatrix}$$

$$= \begin{pmatrix} 43^{ksi} \\ 70.6^{ksi} \\ 11.1^{ksi} \end{pmatrix}$$

④ von Mises (Al is ductile material)

$$\frac{1}{6}\left[(43-70.6)^2 + (43)^2 + (70.6)^2 + 6(11.1)^2\right] = 1389$$

$$\frac{Y^2}{3} = \frac{36^2}{3} = \underline{432 << 1389}$$

⑤

This stress state is definitely $\boxed{\text{NOT SAFE.}}$ by $\approx 3x$

⑥ Alternative?

Steel 5Cr-Mo-V has 3x higher E and 5x higher yield strength $5/3 < 3$ No

Titanium has 1.5x higher E and 3x higher yield $\frac{3}{1.5} \approx 2 < 2$ No

Kevlar has $\approx 1x$ (same) E and 10x higher yield $10 > 3$ Yes