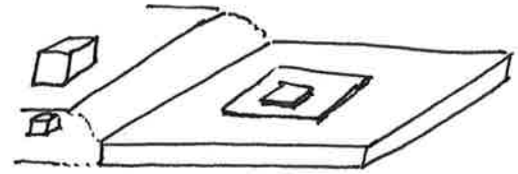
Transversely Isotropic Hookean Meterial

orthotropic

Consider a material with a "strong/stiff" direction and two weaker directions.

wood

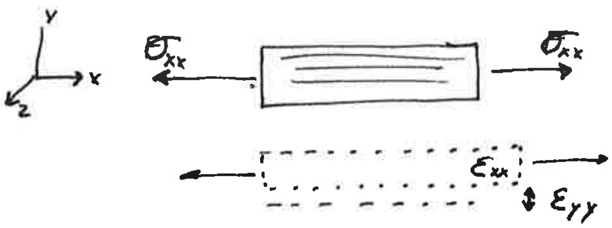
unidirectional composite (e.g. CF tape)



Cold rolled metal (e.g. sheet metal)
flattened grain structure

Perform the same set of stress-strain tests

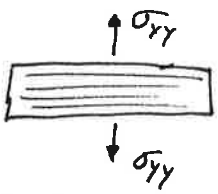
① Uniaxial along x direction



$$\sigma_{xx} = E_x \varepsilon_{xx} \Rightarrow \varepsilon_{xx} = \frac{\sigma_{xx}}{E_x}$$

$$\nu_{xy} = \frac{-\varepsilon_{yy}}{\varepsilon_{xx}}$$

② Uniaxial along y direction

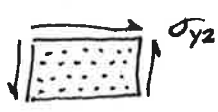


$$E_y = \frac{\sigma_{yy}}{\varepsilon_{yy}} \Rightarrow \varepsilon_{yy} = \frac{\sigma_{yy}}{E_y}$$

$$\nu_{yz} = \frac{-\varepsilon_{zz}}{\varepsilon_{yy}}$$

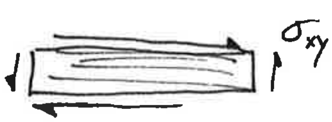
③ Unidirectional along z direction
Same concept as #2

④ Shear in y-z plane



$$\varepsilon_{yz} = \frac{\sigma_{yz}}{G_{yz}}$$

⑤ Shear in x-y direction (and x-z is similar)



$$\varepsilon_{xy} = \frac{\sigma_{xy}}{G_{xy}}$$

Notice that this shear modulus is <u>not</u> the same as the Gyz modulus, since there are fibers in the x-direction!

Summation of strains (similar to previous) gives

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{pmatrix} = \begin{bmatrix} \frac{1}{E_x} & \frac{-\nu_{xy}}{E_x} & \frac{-\nu_{xy}}{E_x} & & & \\ \frac{-\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{-\nu_{yz}}{E_y} & & & \\ \frac{-\nu_{xy}}{E_x} & \frac{-\nu_{yz}}{E_y} & \frac{1}{E_y} & & & \\ & & & \frac{2(1+\nu_{yz})}{E_y} & & \\ & & & & \frac{1}{G_{xy}} & \\ & & & & & \frac{1}{G_{xy}} \end{bmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix}$$

Material properties:

$E_x, E_y, \nu_{xy}, \nu_{yz}, G_{xy}$

Ex: Given Carbon Fiber unidirectional tape ($E_x \approx 25\times10^6$ psi, $E_y \approx 2\times10^6$ psi, $\nu = 0.3$, $G_{xy} \approx 730^{\text{ksi}}$)

Determine the strain given a load in the direction of the tape

$\sigma_{xx} = 150^{\text{ksi}}$ $\quad \sigma_{yy} = 0 = \sigma_{zz}$

$\sigma_{yz} = 2^{\text{ksi}} = \sigma_{xy} \quad \sigma_{xz} = 0$

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E_x} = \frac{150\text{ ksi}}{25\times10^6\text{ psi}} \cdot \frac{1000\text{ psi}}{\text{ksi}} = 0.006$$

$$\varepsilon_{yy} = -\frac{\nu_{xy}}{E_x}\sigma_{xx} = 0.3 \cdot 0.006 = 0.0018$$

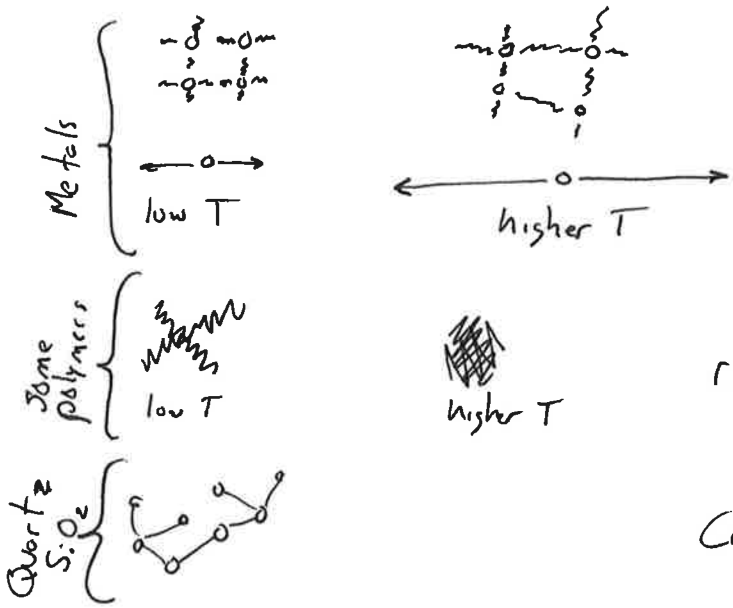
$$\varepsilon_{zz} = 0.0018$$

$$\varepsilon_{yz} = \frac{2(1+0.3)}{2\times10^6\text{ psi}} \cancelto{2\text{ ksi}}{\sigma_{yz}} = 0.0026$$

$$\varepsilon_{xy} = \frac{\sigma_{xy}}{G_{xy}} = \frac{2^{\text{ksi}}}{730^{\text{ksi}}} = 0.00274$$

Thermoelastic Solids

Temperature is a measure of the microscopic vibrational kinetic energy distribution.



$$T = C \cdot \underbrace{\left\langle \frac{mv^2}{2} \right\rangle}_{\text{distribution average}}$$

rubber!

Crystal from Greek κρυος (ice cold)
They thought it was a special warm hard ice (H_2O).
"Quartz found near glaciers in mtns!"

Most materials expand with increasing temperature.

Some Polymers, Crystals, and even H_2O (at certain temps) contract with Temp.

The strain corresponding to a temperature change is (1^{st} order expansion)

$$\varepsilon = \alpha \Delta T$$

Ex: A $10^{in} \times 10^{in} \times 10^{in}$ block of 6061-T6 Al is heated from 0°F to 200°F.
Determine the change in volume.

$$V = 10 \cdot 10 \cdot 10 = 1000^{\,in^3}$$

$$\Delta L_{new} = \Delta T \alpha = 13 \times 10^{-6}/_F \; 200F = 0.0026$$

$$V_{new} = \left(10\left(1 + \Delta T \alpha\right)\right)^3 = (10.026)^3$$

$$= 1007.8^{\,in^3} \approx 0.8\%$$

Not a big deal, right?

Thermal can be a big deal

3D Isotropic Hookean Thermoelastic Material

Thermal Strain is: $\varepsilon_{ii} = \alpha \Delta T$ $\qquad \varepsilon_{ij} = 0$ when $i \neq j$

Strategy: Add strains together again.

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz} \boxed{+ \alpha \Delta T}$$

and ε_{yy}, ε_{zz} similiarly...

Stress to strain

Add to previous 3D isotropic hookean..

$$\boxed{\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{pmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & & & \\ -\nu & 1 & -\nu & & & \\ -\nu & -\nu & 1 & & & \\ & & & 2(1+\nu) & & \\ & & & & 2(1+\nu) & \\ & & & & & 2(1+\nu) \end{bmatrix}\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix} \boxed{+ \alpha \Delta T \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}}$$

Invert [C] to get [D] after an algebra operation

$$(\varepsilon) = [C](\sigma) + \alpha \Delta T (N) \Rightarrow [C](\sigma) = (\varepsilon) - \alpha \Delta T (N)$$

so

$$(\sigma) = C^{-1}(\varepsilon) - \alpha \Delta T [C]^{-1}(N)$$

But $[C]^{-1} = [D]$ and we have [D] from before, so $\alpha \Delta T [C]^{-1} N = \alpha \Delta T [D] N$

$$\alpha \Delta T [C]^{-1}(N) = \alpha \Delta T \begin{bmatrix} 1 & -\nu & -\nu & \cdots \\ -\nu & 1 & -\nu & \\ -\nu & -\nu & 1 & \\ \vdots & & & \ddots \end{bmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ \vdots \end{pmatrix} =$$

$$[C] = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \qquad \det(C) =$$

$$f_{11} = m_{33}\cdot m_{22} - m_{23}^2 = \underline{1-\nu^2} \qquad f_{12} = \nu^2+\nu \qquad f_{13} = \nu^2 + \nu = (1+\nu)\nu$$

$$f_{22} = 1-\nu^2 \qquad f_{23} = \nu^2+\nu = \nu(1+\nu)$$

$$f_{33} = 1-\nu^2$$

$$\det(C) = 1\cdot(1-\nu^2) + -\nu(\nu)(\nu+1) + (-\nu)\nu(1+\nu)$$

$$= (1-\nu^2) - 2\nu^2(\nu+1) = 1-\nu^2 - 2\nu^3 - 2\nu^2 = (2\nu-1)(-1)(\nu+1)^2$$

Terms for

$$\frac{1-\nu^2}{(1-2\nu)(\nu+1)^2} = \frac{(1-\nu)(1+\nu)}{(1-2\nu)(1+\nu)(1+\nu)} = \frac{1-\nu}{(1-2\nu)(1+\nu)}$$

$$\frac{\nu^2+\nu}{(1-2\nu)(\nu+1)^2} = \frac{\nu(1+\nu)}{(1-2\nu)(\nu+1)(\nu+1)} = \frac{\nu}{(1-2\nu)(\nu+1)}$$

$$[D] = \frac{E}{(1-2\nu)(1+\nu)}\begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix}$$

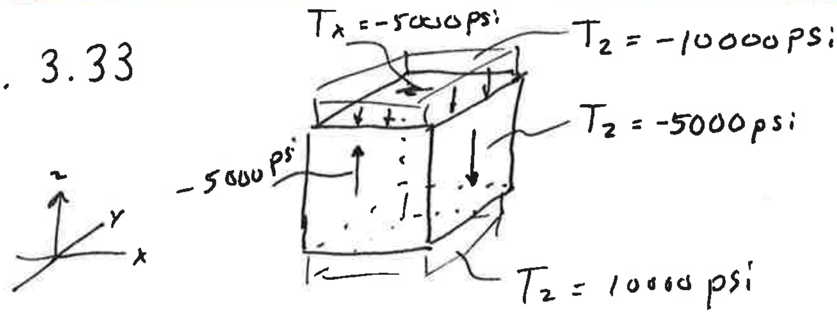
$$\alpha \Delta T [C]^{-1}(N) = \alpha\Delta T[D]N = \frac{\alpha\Delta T E}{(1-2\nu)(1+\nu)}\begin{pmatrix} 1-\nu+\nu+\nu \\ 1-\nu+\nu+\nu \\ 1-\nu+\nu+\nu \end{pmatrix} \begin{pmatrix} 1+\nu \\ 1+\nu \\ 1+\nu \end{pmatrix}$$

$$= \frac{\alpha\Delta T E}{(1-2\nu)}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Strain to Stress

$$\boxed{\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{xz}\\ \sigma_{xy}\end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & & \\ \nu & \nu & 1-\nu & & & \\ & & & \frac{1-2\nu}{2} & & \\ & & & & \frac{1-2\nu}{2} & \\ & & & & & \frac{1-2\nu}{2}\end{bmatrix}\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{yz}\\ \varepsilon_{xz}\\ \varepsilon_{xy}\end{pmatrix} - \frac{\alpha\,\Delta T\,E}{1-2\nu}\begin{pmatrix}1\\1\\1\\0\\0\\0\end{pmatrix}}$$

Ex. 3.33



$\alpha = +13 \times 10^{-6}/F$

$\nu = 0.30$

$\Delta T = 100\,F$

$K = 10 \times 10^{6}$ psi

Stress

$\sigma_{zz} = 10^{\text{ksi}}$ $\quad \sigma_{xx} = 0$ $\quad \sigma_{yy} = 0$ $\quad \sigma_{xy} = 0$ $\quad \sigma_{xz} = -5^{\text{ksi}}$

Bulk Mod'

$$K \equiv \frac{E}{3(1-2\nu)} \Rightarrow E = K \cdot 3 \cdot (1-2\nu) = 10 \times 10^{6}\,\text{psi} \cdot 3 \cdot (1 - 2 \cdot 0.3)$$

$$= 12 \times 10^{6}\,\text{psi}$$

Strains

$$\varepsilon_{xx} = \frac{\cancelto{0}{\sigma_{xx}} - \nu \cancelto{0}{\sigma_{yy}} - \nu\sigma_{zz}}{E} = \frac{-0.3 \cdot 10^{\text{ksi}}}{12 \times 10^{6}\,\text{psi}} \cdot \frac{1000\,\text{psi}}{\text{ksi}} = \boxed{0.00025}$$

$$\varepsilon_{yy} = 0.00025$$

$$\varepsilon_{zz} = \frac{\cdots\ \sigma_{zz}}{E} = \boxed{0.00083}$$

$$\varepsilon_{xz} = \frac{2(1+\nu)\sigma_{xz}}{E} = \frac{2(1.3)(-5^{\text{ksi}})}{12 \times 10^{6}\,\text{psi}} = \boxed{0.0011}$$

principal stresses

$$\text{eigval}(\sigma) = \text{eigval}\begin{pmatrix}0 & 0 & -5\\ 0 & 0 & 0\\ -5 & 0 & 10\end{pmatrix} = \boxed{12.0^{\text{ksi}},\ -2.0^{\text{ksi}},\ 0.0^{\text{ksi}}}$$

$$\sigma_{s_{max}} = \frac{12-(-2)}{2} = \frac{14}{2} = \boxed{7^{\text{ksi}}}$$

Composites

Nomenclature and Terminology is often derived from textile industry.

Single fiber  filament (1)

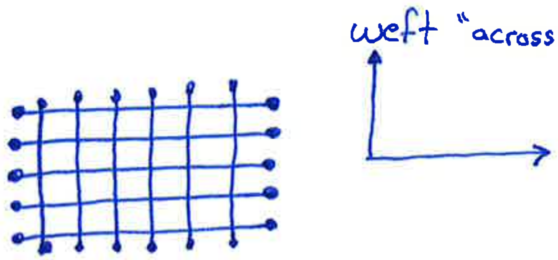
Multiple filaments  strand (100)

Multiple strands  tow (3000)

twisted strands  yarn

$\underbrace{\text{twisted strands} \quad \text{yarn}}_{\text{Not as common in aero}}$

Woven tow
"Cloth"



Chopped Strand Mat
"CSM"

Usually glass fibers chopped and then randomly oriented in a "mat" held together with a binder "temp glue"

Tapes

Can be uni or woven

Pre-preg

the matrix (in an uncured state) is applied to the material

Usually stored in a freezer, limited shelf life, Wonderful to use!
precise fiber to matrix ratio

Alternative ways to determine eigenvalues of a 3x3 $[\sigma]$

TI-84

youtu.be/n9_39MwNKGc

solve(det(x(Identy(3) - M))
good guesses

Numerical Solver

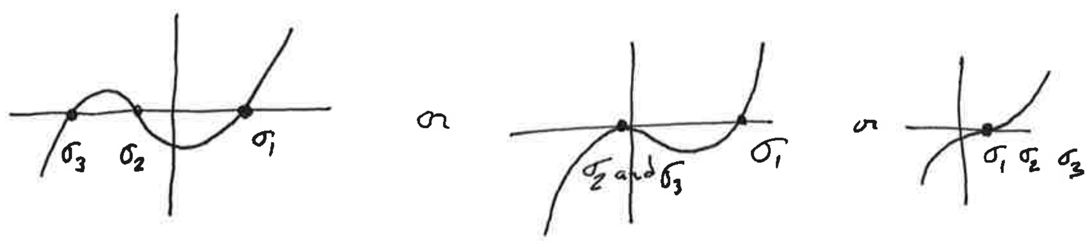
$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{zz} & \sigma_{zx} \\ \sigma_{zx} & \sigma_{xx} \end{vmatrix}$$

$$I_3 = \det(\sigma) = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}$$

$$\boxed{-\sigma_p^3 + I_1\sigma_p^2 - I_2\sigma_p + I_3 = 0}$$

plot ↑ as $y(x) = -x^3 + I_1x^2 - I_2x + I_3$



Online

www.bluebit.gr/matrix-calculator

Matlab