Airy Stress Functions

A function Φ (phi) where the stresses in 2D are:

$$\boxed{\sigma_x = \frac{d^2\phi}{dy^2} \qquad \sigma_y = \frac{d^2\phi}{dx^2} \qquad \sigma_{xy} = -\frac{d^2\phi}{dxdy}}$$

Notice that this is similar to the potential function in aerodynamics!

$$\boxed{\nabla^4\phi = 0}$$

Remember from stress equilibrium equations that

$$\frac{d\sigma_x}{dx} + \frac{d\sigma_{xy}}{dy} + \cancel{X}^{0} = 0$$

and

$$\frac{d\sigma_y}{dy} + \frac{d\sigma_{xy}}{dx} + \cancel{Y}^{0} = 0$$

plug in above $\longrightarrow$

$$\frac{d^3(\phi)}{dx\,dy^2} + \frac{-d^3\phi}{dy^2dx} = 0 \quad \checkmark \text{ true}$$

$$\frac{d}{dy}\left(\frac{d^2\phi}{dx^2}\right) + \frac{d}{dx}\left(\frac{-d^2\phi}{dxdy}\right) = 0 \quad \checkmark \text{ true}$$

So the Airy definitions fit the stress equilibrium equations

Remember that we found earlier that

$$\varepsilon_x = \frac{du}{dx}$$

the x-direction strain is how much x deflection occurs in the x direction.

$$\varepsilon_y = \frac{dv}{dy}$$

and

$$\varepsilon_{xy} = \frac{du}{dy} + \frac{dv}{dx}$$

Add a "magic" operation.

1. Take the 2nd derivative of ε_x wrt y
2. Add the 2nd derivative of ε_y wrt x $\Rightarrow \frac{d^2\varepsilon_x}{dy^2} + \frac{d^2\varepsilon_y}{dx^2}$
3. Expand $\frac{d^2\varepsilon_x}{dy^2} = \frac{d^2}{dy^2}\left(\frac{du}{dx}\right)$ and $\frac{d^2}{dx^2}\left(\frac{dv}{dy}\right)$.
4. Take the 2nd derivative of ε_{xy} wrt x and y.

$$\frac{d^2}{dx\,dy}(\varepsilon_{xy}) = \frac{d^2}{dxdy}\left(\frac{du}{dy} + \frac{dv}{dx}\right) = \frac{d^3u}{dx\,dy^2} + \frac{d^3v}{dx^2dy}$$

5. Notice that

1 + 2 = 4

$$\frac{d^2\varepsilon_x}{dy^2} + \frac{d^2\varepsilon_y}{dx^2} = \frac{d^3u}{dy^2dx} + \frac{d^3v}{dx^2dy} = \frac{d^2}{dxdy}(\varepsilon_{xy})$$

The strain compatibility equation (2D) is

$$\boxed{\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \varepsilon_{xy}}{\partial x\, \partial y}}$$

What is this telling us?
No cracks or folds

But we have the 2D isotropic Hookes Law/material (Lesson 8)

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} = \frac{1}{E}\begin{pmatrix} 1 & -\nu & \\ -\nu & 1 & \\ & & 2(1+\nu) \end{pmatrix}\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}$$

So that

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu \sigma_{yy}}{E} \qquad \varepsilon_{yy} = \frac{-\nu \sigma_{xx}}{E} + \frac{\sigma_{yy}}{E}$$

and

$$\varepsilon_{xy} = 2(1+\nu)\,\sigma_{xy}$$

- plug these into the strain compatibility eqs (above)

$$\frac{\partial}{\partial y^2}\left(\frac{\sigma_{xx}}{E} - \frac{\nu}{E}\sigma_{yy}\right) + \frac{\partial}{\partial x^2}\left(-\frac{\nu\sigma_{xx}}{E} + \frac{\sigma_{yy}}{E}\right) = \frac{\partial^2}{\partial x\partial y}\left(\frac{2(1+\nu)\,\sigma_{xy}}{E}\right)$$

- pull out the E and substitute $\sigma_x = \frac{d^2\phi}{dy^2}$, $\sigma_y = \frac{d^2\phi}{dx^2}$, $\sigma_{xy} = \frac{d^2\phi}{dx\,dy}$

$$\frac{d^2}{dy^2}\left(\frac{d^2\phi}{dy^2} - \nu\frac{d^2\phi}{dx^2}\right) + \frac{d^2}{dx^2}\left(-\nu\frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dx^2}\right) = \frac{d^2}{dx\,dy}\left(\underbrace{\frac{E\;2(1+\nu)}{E}}_{\text{Recognize this?}}\;\frac{-d^2\phi}{dx\,dy}\right)$$

$$G = \frac{1}{2}\frac{E}{1+\nu}$$

thus,

$$\frac{2E(1+\nu)}{E} = 2(1+\nu)$$

And

$$2(1+\nu)(\cdots) = (1+\nu)(\cdots + \cdots)$$

- Expand out

$$\frac{d^4}{dy^4}\phi - \frac{d^4(\nu\phi)}{dx^2dy^2} - \nu\frac{d^4\phi}{dx^2dy^2} + \frac{d^4\phi}{dx^4} = (1+\nu)\left(-\frac{d^4\phi}{dx^2dy^2} + \frac{-d^4\phi}{dx^2dy^2}\right)$$

cancel out

$$\boxed{\frac{d^4\phi}{dy^4} + \frac{d^4\phi}{dx^4} + 2\frac{d^4\phi}{dx^2dy^2} = 0}$$

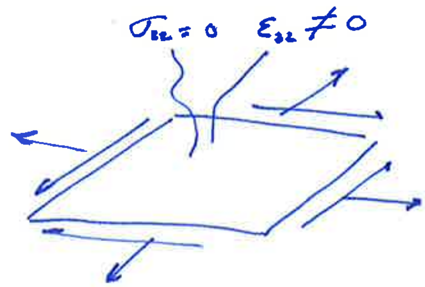
This is the biharmonic equation $\boxed{\nabla^4\phi = 0}$

$$\nabla^2(\underbrace{\nabla^2\phi}_{\sigma_x+\sigma_y\,!}) = \nabla^4\phi$$

Alternatively,

$$\boxed{\nabla^2(\sigma_x + \sigma_y) = 0}$$

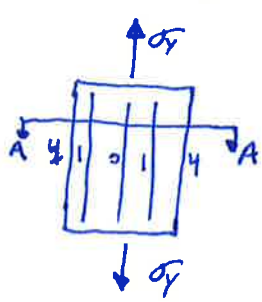
Notice that this derivation is for plane stress.

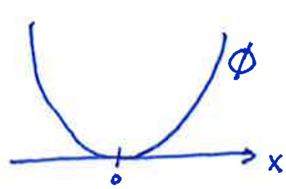


Ex: $\phi = A$ $\qquad \sigma_x = \frac{d^2\phi}{dy} = 0 \qquad \sigma_y = 0$ too $\qquad$ <u>boring solution</u>

Ex: $\phi = Ax$ $\qquad \sigma_x = \frac{d^2\phi}{dy} = 0 \qquad \sigma_y = \frac{d}{dx}\left(\frac{d}{dx}Ax\right) = \frac{d}{dx}A = 0$ $\qquad$ <u>boring</u>

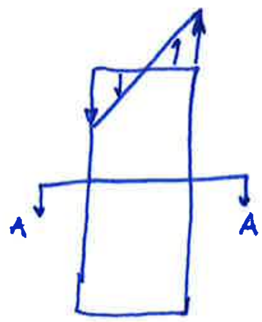
Ex: $\phi = Ax^2$ $\qquad \sigma_x = 0 \qquad \sigma_y = \frac{d^2}{dx}(Ax^2) = 2A$

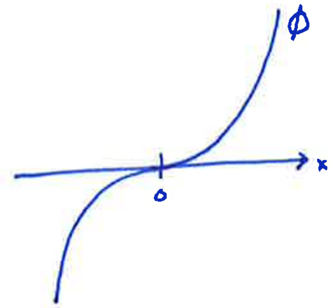




Ex: $\phi = Ax^3$

$\sigma_x = 0 \qquad \sigma_y = \frac{d^2}{dx^2}(Ax^3) = 6Ax \qquad \sigma_{xy} = 0$





Ex: $\phi = Ax^4$

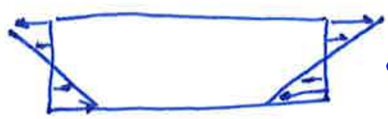
$\sigma_x = 0 \qquad \sigma_y = 12Ax^2 \qquad \sigma_{xy} = 0$

But, is $\nabla^4\phi$ satisfied?

$$\nabla^4\phi = \cancelto{0}{\frac{d^4\phi}{dy^4}} + \cancelto{12A}{\frac{d^4\phi}{dx^4}} + 2\cancelto{0}{\frac{d^4\phi}{dx^2dy^2}} = 12A \neq 0$$

Ex: $\phi = Ay^3$ Verify $\nabla^4\phi = 0$ ✓ by inspection

$\sigma_x = \frac{d^2}{dy^2}(Ay^3) = 6Ay, \quad \sigma_y = 0, \quad \sigma_{xy} = 0$



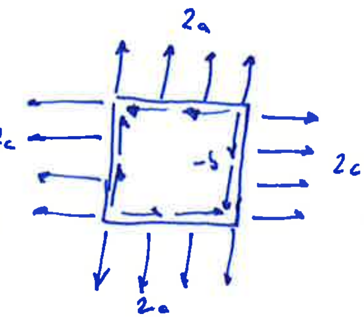
$\sigma_x = 6Ay$ Beam bending!

Ex:

If any two solutions fit the Boundary Conditions and Airy stress $\nabla^4\phi = 0$ requirements then any combination does too

$\phi = ax^2 + bxy + cy^2 \qquad \nabla^4\phi = 0$ ✓

$\sigma_x = \frac{d}{dy^2}(\phi) = 2c \qquad \sigma_y = 2a \qquad \sigma_{xy} = -b$



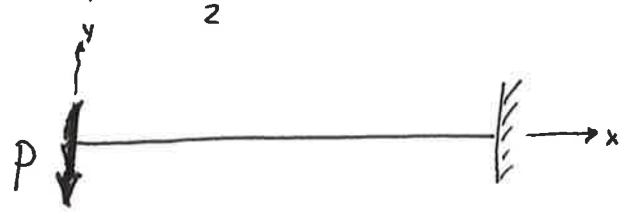
Ex: Cantilever Beam

$$\phi = A\left(xy^3 - \frac{3}{4}xyh^2\right)$$

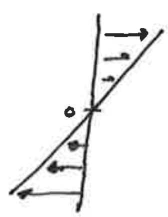
$$\sigma_x = \frac{d^2}{dy^2}\phi = 6Axy \qquad \sigma_y = \frac{d^2}{dx^2}\phi = 0$$

$$\sigma_{xy} = \frac{-d^2\phi}{dx\,dy} = 3A\left(\frac{h^2}{4} - y^2\right)$$

This is exactly a cantilever beam of height h and width b loaded by a force $P = \frac{Abh^3}{2}$

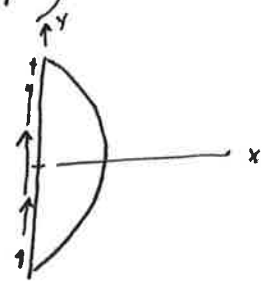


The x stress $\sigma_x = 6Axy$



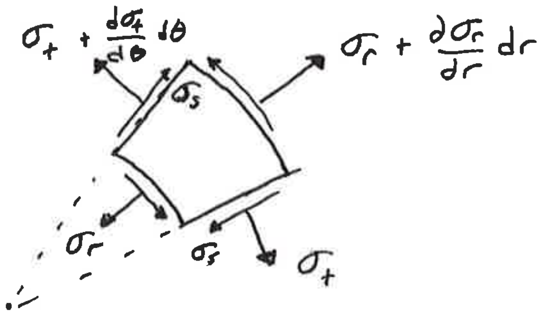
The shear stress

$$\sigma_{xy} = 3A\left(\frac{h^2}{4} - y^2\right)$$



parabolic distribution of shear stress

Polar Coordinates



Summation of forces in radial and tangential direction

$$\boxed{\frac{\partial \sigma_t}{\partial \theta} + 2\sigma_s + r\frac{\partial \sigma_s}{dr} = 0}$$

$$\boxed{\sigma_r - \sigma_t + r\frac{\partial \sigma_r}{dr} + \frac{\partial \sigma_s}{d\theta} = 0}$$

In this case, the Airy stress function still exists

$$\boxed{\nabla^4 \phi = 0}$$

But $\nabla^2 \phi$ is more complicated $= \frac{\partial^2 \phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + \frac{1}{r^2}\frac{d^2\phi}{d\theta^2} = 0$

And $\nabla^4 \phi$ is $\nabla^2(\nabla^2 \phi)$ is much more complicated

$\phi = \phi(r, \theta)$ in 2D

Ex: $\phi = r^2 \theta^2$

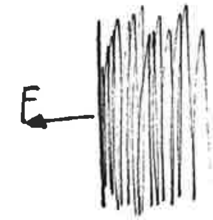
$$\nabla^2 \phi = \frac{d^2(r^2\theta^2)}{dr^2} + \frac{1}{r}\frac{\partial(r^2\theta^2)}{dr} + \frac{1}{r^2}\frac{d}{d\theta^2}(r^2\theta^2)$$

$$2\theta^2 + \frac{1}{r}2r\theta^2 + \frac{1}{r^2}2r^2 = 2\theta^2 + 2\theta^2 + 2$$

$$\nabla^2(\nabla^2 \theta) = \frac{d^2}{dr^2}\cancelto{0}{(2\theta^2 + 2\theta^2 + 2)} + \cancelto{0}{\frac{1}{r}\frac{d(\ldots)}{dr}} + \frac{1}{r^2}\frac{d}{d\theta^2}(4\theta^2 + 2) = \boxed{\frac{8}{r^2} \neq 0}$$

Not an ASF

Concentrated Load on the edge of a plate



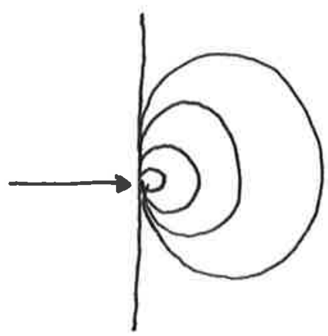
$$\phi = Cr\theta \sin\theta \qquad \nabla^4\phi = 0 \ \checkmark$$

$$\boxed{\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}} \qquad \boxed{\sigma_t = \frac{\partial^2 \phi}{\partial r^2}} \qquad \boxed{\sigma_s = -\frac{\partial}{\partial r}\left(\frac{\partial \phi}{r\,\partial \theta}\right)}$$

Applied to $\phi = Cr\theta\sin\theta$

$$\sigma_r = 2C\frac{\cos\theta}{r} \qquad \sigma_t = 0 \qquad \sigma_s = 0$$

The point load on the edge of a plate gives only radial stresses!



Clearly, the stress at $r = 0 \to \infty$ since $\sigma_r(0) = 2C\frac{\cos\theta}{0}$

So there is plastic deformation near the point load, but this quickly goes to the far field equation of

$$\sigma_r(r) = 2C\frac{\cos\theta}{r}$$

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Good luck on the exam.

I will send the formula sheet soon (via email).