# Advanced Beam Theory

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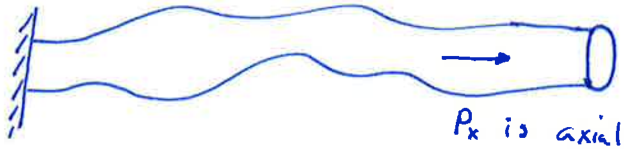
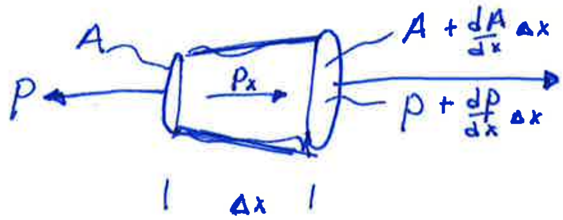
## Allen and Haisler Chap 4

## Uniaxial bar



$P_x$ is axial load per unit length

Alternative derivation.




① $\sum F_x = 0 = \underbrace{-P}_{\text{left}} + \underbrace{P + \frac{dP}{dx}\Delta x}_{\text{right}} + \underbrace{P_x \Delta x}_{\text{applied load}}$

$$= \frac{dP}{dx}\Delta x + P_x \Delta x = 0 \quad \Rightarrow \quad P_x = -\frac{dP}{dx}$$

② Isotropic material (1D)

$$\sigma = E\varepsilon$$

③ stress and strain

$$\sigma = \frac{P}{A} = E\varepsilon \quad \text{and} \quad \varepsilon \equiv \frac{du}{dx}$$

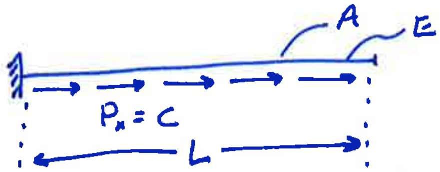
Thus $P = AE\frac{du}{dx}$

④ substitute into $\frac{dP}{dx} = -P_x$

$$\boxed{\frac{d}{dx}\left(AE\frac{du}{dx}\right) = -P_x}$$

Easier derivation than in the book.
No traction vectors involved
No surface integrals or Green's theorem

Ex: Constant Axial load per unit length




① $P$ diagram (force)

$\frac{dP}{dx} = -p_x = -c$ integrate $P = -cx + K_1$ and apply BC $P(L) = 0 \Rightarrow K_1 = cL$

$P = -cx + cL$


② Gov Eq.

$\frac{d}{dx}\left(AE\frac{du}{dx}\right) = -p_x$ since AE is constant $\frac{d^2u}{dx^2} = -\frac{p_x}{AE}$

③ Integrate

- $d\left(\frac{du}{dx}\right) = -\frac{p_x}{AE}\,dx$ integrate (definite) $\int d\frac{du}{dx} = \int_{\varepsilon_0}^{\varepsilon(x)} d\varepsilon = \int_0^x \frac{-p_x}{AE}\,dx$ ($\frac{du}{dx} \to \varepsilon$)

$\varepsilon(x) - \varepsilon_0 = -\frac{p_x}{AE}x \Rightarrow \varepsilon(x) = \varepsilon_0 - \frac{p_x}{AE}x$

- $\varepsilon = \frac{du}{dx} = \varepsilon_0 - \frac{p_x}{AE}$ integrate (definite)

$$\int_{u_0}^{u(x)} du = \int_0^x \left(\varepsilon_0 - \frac{p_x}{AE}x\right) dx$$

$$u(x) = u_0 + \varepsilon_0 x - \frac{p_x x^2}{AE\,2}$$

④ Boundary Conditions

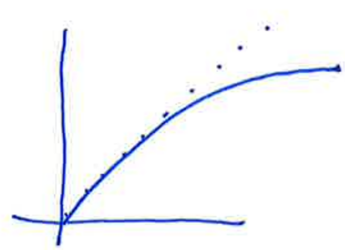

$u_0 = 0$ fixed into the wall

$\varepsilon_0$ is the strain at the wall.

$P(x=0) = cL$ $\quad \varepsilon_0 = \frac{\sigma}{E} = \frac{P}{AE} = \frac{cL}{AE}$

$$\boxed{u(x) = \frac{cL}{AE}x - \frac{cx^2}{2AE}} = \frac{p_x L}{AE}x - \frac{p_x x^2}{2AE} = \frac{p_x}{AE}\left(Lx - x^2\right)$$

$\varepsilon = \frac{du}{dx}$ or $\varepsilon(x) = \varepsilon_0 - \frac{p_x}{AE}x = \frac{p_x L}{AE} - \frac{p_x}{AE}x$



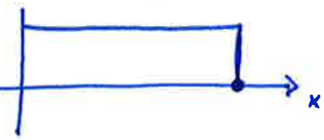
Ex: End Load


① $p$ diagram

$$\frac{dp}{dx} = -p_x = -P\,\delta(x-L) \quad \text{integrate} \quad \int_{p_0}^{p(x)} dp = -\int_0^x P\,\delta(x-L)$$

$$p(x) - p_0 = -P\,H(x-L)$$

$$p(x) = p_0 - P\,H(x-L) = P(1 - H(x-L))$$




Some Math

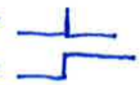
$\delta$ = delta function

$\int \delta = H$

$H \equiv$ Heaviside function

$\delta =$

$H =$



② Gov Equ

$$\frac{d}{dx}\left(AE\,\frac{du}{dx}\right) = -p_x \Rightarrow \frac{d^2u}{dx^2} = \frac{-p_x}{AE} = \frac{-P\,\delta(x-L)}{AE}$$

③ Integrate $\quad \varepsilon = \frac{du}{dx}$

$$\int \frac{d}{dx}(\varepsilon)\,dx = \int \frac{-P}{AE}\,\delta(x-L)\,dx \Rightarrow \varepsilon(x) - \varepsilon_0 = \frac{-P}{AE}\left[H(x-L) - \underbrace{\cancel{H(0-L)}}_{\text{always } 0}\right]$$

$$\varepsilon(x) = \varepsilon_0 + \frac{-P}{AE}\,H(x-L) \quad \text{within the domain } 0 \le x < L$$

$$\varepsilon(x) = \varepsilon_0$$

④ Integrate

$$\varepsilon(x) = \frac{du}{dx} \Rightarrow \int_{u_0}^{u(x)} du = \int_0^x \varepsilon_0 - \frac{P}{AE}\,H(x-L)\;dx$$

$$u(x) - u_0 = \varepsilon_0 x - \frac{P}{AE}\,H(x-L)(x-L) = \varepsilon_0 x \quad \text{in} \quad 0 \le x < L$$

$$u(x) = u_0 + \varepsilon_0 x$$

⑤ BC.

$u_0 = 0$

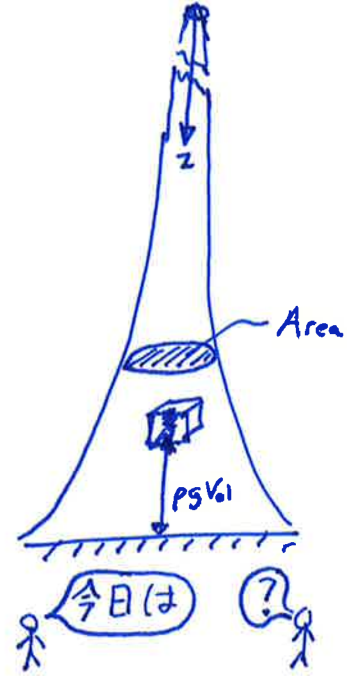
$\varepsilon_0 = \frac{p_0}{AE} = \frac{P}{AE}$

$$\boxed{u(x) = \frac{P}{AE}\,x}$$

$$\boxed{\varepsilon(x) = \frac{P}{AE}}$$

<u>As expected</u>

Ex: How tall can you build a tower?




- Gravity


- Material

  Operate <u>just</u> at the yield stress.

- Geometry

  Does round or square matter?

① Stress

$$\sigma_z = \varepsilon_z E = E \frac{du}{dz}$$

② Gravity

$$p_x = \rho g A$$

② Gov Eqn

$$\frac{d}{dz}\left(AE\frac{du}{dz}\right) = -p_x = -\rho g A \quad \text{or} \quad \frac{dA}{dz} E \frac{\sigma_z}{E} = -\rho g A$$

non constant ↗ ↑ material ↖ $\frac{\sigma_z}{E}$

$$\underline{\frac{dA}{dz} = \frac{-\rho g A}{\sigma_z}}$$ Differential equation

③ Solve ODE

$$\frac{dA}{A} = \left(-\frac{\rho g}{\sigma_z}\right) dz \Rightarrow \underbrace{\ln A - \ln A_0}_{\ln \frac{A}{A_0}} = \left(-\frac{\rho g}{\sigma_z}\right)\left(z - \cancel{z_0}\right)$$ start at top

$$\ln \frac{A}{A_0} = -\frac{\rho g}{\sigma_z} z$$

↖ Notice that we need to refine the problem statement since $\ln 0 \to -\infty$
Make the tower useful ~ 1 ft² at top.

④ Solve for area

$$\frac{A}{A_0} = e^{-\frac{\rho g}{\sigma_z} z}$$

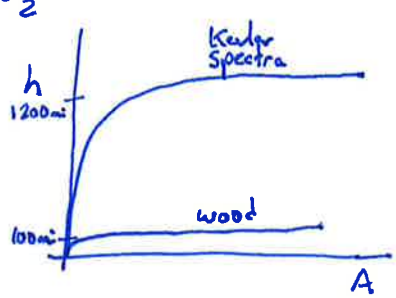
$$A = A_0 e^{-\frac{\rho g}{\sigma_z} z}$$

exponential shape

$$A = e^{-\frac{\rho g}{\sigma_z} z} =$$

From the aero materials, Kevlar and HDPE were about $\frac{10000\ \text{ksi}}{\text{lbf}/\text{in}^3} \approx \frac{320000\ \text{ksi}}{\text{slug}/\text{in}^3}$

$$A = e^{1.2 \times 10^{-6} z}$$




$$\frac{\rho g}{\sigma_z} = \frac{\text{slugs}}{320000\ \text{ksi}\ \text{in}^3}\Big|\frac{32.174\ \text{ft}}{\text{s}^2}\Big|\frac{\text{ft}}{}\Big|\frac{\text{lbf}\ \text{s}^2}{\text{slug}\ \text{ft}}\Big|\frac{\text{ksi}\ \text{in}^2}{1000\ \text{lbf}}\Big|\frac{12\ \text{in}}{\text{ft}}$$

$$= \frac{\cancel{\quad} \cdot 12}{10000 \cdot 1000} = \cancel{\quad} \text{ mil } 2\,[\text{ft}]$$

$$= 1.2 \times 10^{-6}$$