How to find the centroid and I_{yy} I_{zz} moments of inertia of an advanced beam.

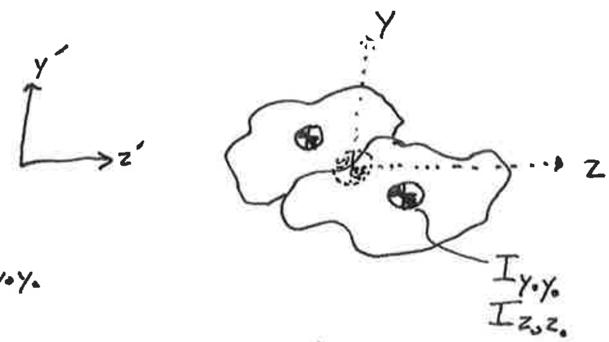
① Centroid

- Decompose into parts
- With respect to <u>any</u> coordinate system origin
- $\sum A_i$ sum areas of each part

$\sum z_i A_i$ sum (distance from centroid of each part) $\cdot$ (area of each part)

Centroid is at $\dfrac{\sum z_i A_i}{\sum A_i}$

② Moments of inertia

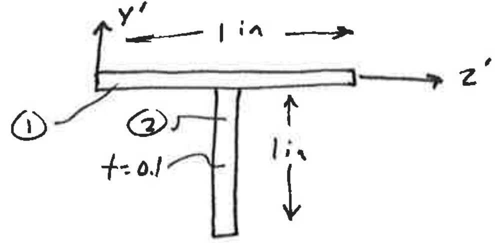


- With respect to $y'z'$
- Determine $\sum y_i'^2 A_i$ and $\sum I_{y_0y_0}$
- $I_{y'y'} = \sum I_{y_0y_{0_i}} + \sum z_i'^2 A_i$
- $I_{yy} = I_{y'y'} - \bar{z}'^2 A$ shift to centroid

Notice that this is <u>much</u> easier than shifting each part to the centroid (even if the math is eventually identical)

See example 4.1 (critical to understand)

Ex: T beam



$I_{①yy} = \frac{1}{12}bh^3 = \frac{1}{12}\,0.1\cdot 1^3 = 0.00833$

$I_{①zz} = \frac{1}{12}\,1\,0.1^3 = 8.3\times10^{-5}$

① Centroid

Part	Area	z'	y'	$z'A$	$y'A$
1	0.1	0.5	0	0.05	0
2	0.1	0.5	-0.5	0.05	-0.05
	0.2			0.1	-0.05

$\bar{z}' = \frac{0.1}{0.2} = 0.5$

$\bar{y}' = \frac{-0.05}{0.2} = -0.25$

② MoI

distance to centroid of part from y'

Part	A	$\bar{y}_i'$	$\bar{z}_i'$	$\bar{y}_i'^2 A_i$	$\bar{z}_i'^2 A_i$	$I_{y_0y_0}$	$I_{z_0z_0}$
1	0.1	0	0.5	0	0.025	0.00833	8.3×10^{-5}
2	0.1	-0.25	0.5	0.00625	0.025	8.3×10^{-5}	0.00833
				0.00625	0.05	0.00842	0.00842

$I_{y'y'} = \Sigma I_{y_0y_0} + \Sigma z_i^2 A_i = 0.00842 + 0.05 = 0.05842$

$I_{z'z'} = \Sigma I_{z_0z_0} + \Sigma y_i^2 A_i = 0.00842 + 0.00625 = 0.01467$

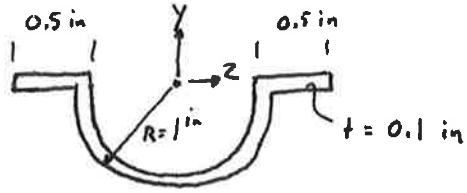
$I_{y'z'} = \Sigma I_{y_0z_0} + \Sigma yzA = 0$

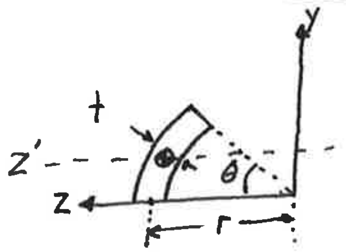
Shift to centroid of beam

$I_{yy} = I_{y'y'} - \bar{z}'^2 A = 0.05842 - 0.5^2\cdot 0.2 = 0.00842$

$I_{zz} = I_{z'z'} - \bar{y}'^2 A = 0.01467 - 0.25^2\cdot 0.2 = 0.00217$

Ex: U channel





Write in your book (p 500)

$$A = rt\theta$$

$$z' = \frac{r}{\theta}(1-\cos\theta)$$

$$I_{zz} = r^3 t\left(\frac{\theta - \sin\theta\cos\theta}{2}\right)$$

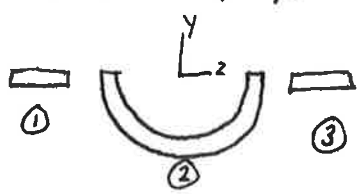
$$I_{z'z'} = r^3 t\left(\frac{\theta - \sin\theta\cos\theta}{2}\right) - \frac{r^3 t}{\theta}(1-\cos\theta)^2$$

θ is in radians

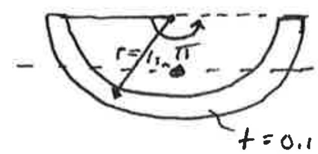
r is median line (i.e. half thickness)

• Find $\bar{y}, \bar{z}$, and I_{yy}, I_{zz}, I_{yz}

① Centroid



part ②



$$z' = \frac{1\text{ in}}{3.1415}(1-\cos\pi)$$

$$= 2/\pi = 0.63662$$

$$A = 1 \,|\, 0.1 \,|\, 3.14 = 0.3141$$

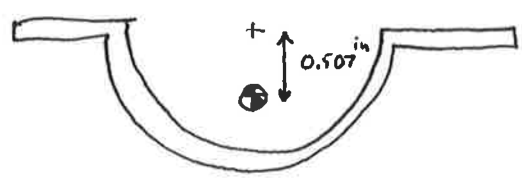
• $\bar{Y}$

part	Area	$\cancel{y}'$	$\cancel{y}' \cdot$ Area
1	0.04	0	0
2	0.314159	-0.63662	-0.2
3	0.04	0	0
Σ	0.3942		Σ -0.2

$$\cancel{y}' = \frac{\Sigma z'A}{\Sigma A} = \frac{-0.2}{0.3942} = -0.507$$

• $\bar{z}$ By inspection = 0

part	Area	z'	$z'A$
1	0.04	0	0
2	0.31415	0	0
3	0.04	0	0
	0.3942		0



$$\bar{z} = \frac{0}{0.3942} = 0$$

② MoI

part	Area	y'	z'	$y'^2 A$	$z'^2 A$	$I_{y_0 y_0}$	$I_{z_0 z_0}$
1	0.04	0	1.2	0	0.0576	0.000533	≈ 0
2	0.314	-0.636	0	0.127	0	0.15708	0.15708
3	0.04	0	-1.2	0	0.0576	0.000533	≈ 0
					0.1152		

$I_{y'y'} = 0.158 + 0.1152 = 0.2723$

$I_{z'z'} = 0.15708 + 0.127 = 0.284$

shift to beam centroid

$I_{yy} = 0.2723 - 0^2 \cdot 0.3942 = 0.2723$

$I_{zz} = 0.284 - 0.507^2 \cdot 0.3942 = 0.1827$

Rotational Transform of M.I

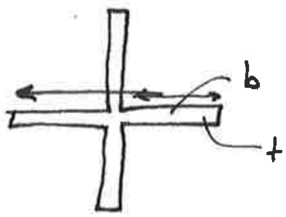
$$I_{xx}^* = \frac{I_{xx}+I_{yy}}{2} + \frac{I_{xx}-I_{yy}}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{yy}^* = \frac{I_{xx}+I_{yy}}{2} - \frac{I_{xx}-I_{yy}}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{xy}^* = \frac{I_{xx}-I_{yy}}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

where have you seen this before?!?...

Ex:



Identical legs.

Is there a "weak" direction of bending?

① By symmetry $I_{yy} = I_{zz} \approx \frac{1}{12} t b^3$

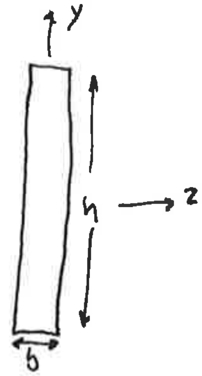
② Rotate by θ

$$I_{yy}^* = \underbrace{\cancel{\frac{I_{yy}+I_{zz}}{2}}}_{I_{yy}=\frac{1}{12}tb^3} + \underbrace{\cancel{\frac{I_{yy}-I_{zz}}{2}}}_{0}\cos 2\theta - \underbrace{\cancel{I_{xy}}}_{0}\sin 2\theta = \frac{1}{12}tb^3 = I_{yy}$$

same for I_{zz}^* and $I_{xy}^* = 0$

Same bending stiffness regardless of rotation

Ex: Beam



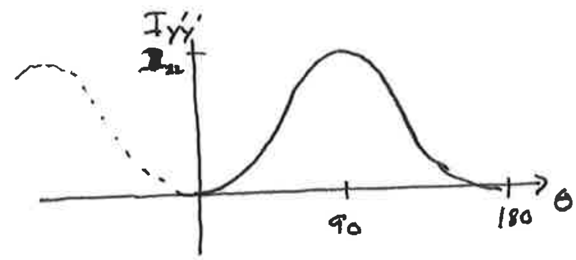
$$I_{yy} = \frac{1}{12} h b^3 \qquad \Rightarrow \qquad \frac{I_{zz}}{I_{yy}} = \frac{\frac{1}{12} b h^3}{\frac{1}{12} h b^3} = \frac{h^2}{b^2}$$

$$I_{zz} = \frac{1}{12} b h^3 \qquad\qquad = \left(\frac{h}{b}\right)^2$$

What is the M.I wrt twist? Assume $\frac{h}{b} \gg 1 \approx 10$

$$I_{y'y'} = \underbrace{\frac{I_{yy} + I_{zz}}{2}}_{\frac{I_{zz}}{2}} + \underbrace{\frac{I_{yy} - I_{zz}}{2} \cos 2\theta}_{-\frac{I_{zz}}{2} \cos 2\theta} - \underbrace{\cancelto{0}{I_{xy}} \sin 2\theta}_{0}$$

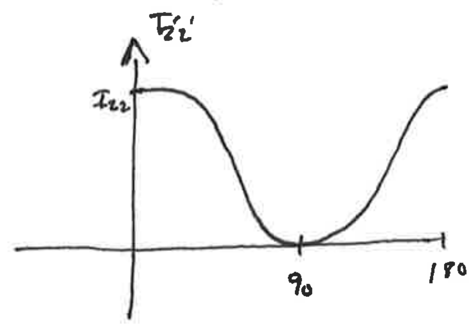
$$= \frac{I_{zz}}{2}\left(1 - \cos 2\theta\right)$$



$$I_{z'z'} = \frac{I_{yy} + I_{zz}}{2} - \frac{I_{yy} - I_{zz}}{2} \cos 2\theta + \cancelto{0}{I_{yz}} \sin 2\theta$$

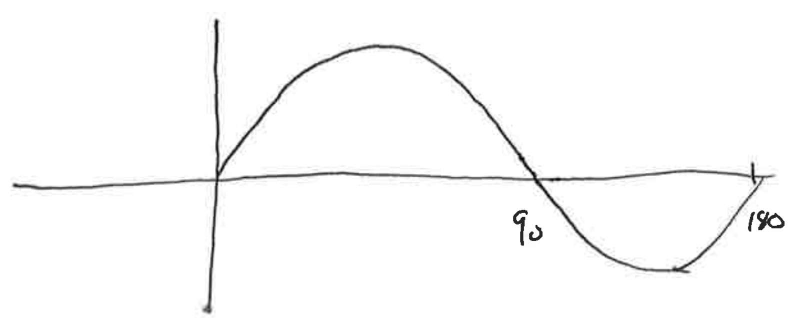
$$\approx \frac{I_{zz}}{2} + \frac{I_{zz}}{2} \cos 2\theta =$$

$$= \frac{I_{zz}}{2}\left(1 + \cos 2\theta\right)$$



$$I_{y'z'} = \frac{I_{yy} - I_{zz}}{2} \sin 2\theta + \cancelto{0}{I_{yz}} \cos 2\theta$$

$$\approx -\frac{I_{zz}}{2} \sin 2\theta$$

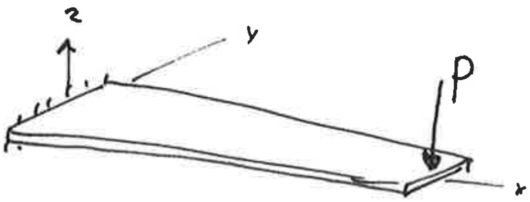


$h/b > 1$ ✓

If a beam twists during loading, the M.I can decrease

Twist Bend Buckling of beams

from Advanced Strength of Materials
Den Hartog



Low I_{yy}
More bending

Compare

High I_{yy}
less bending

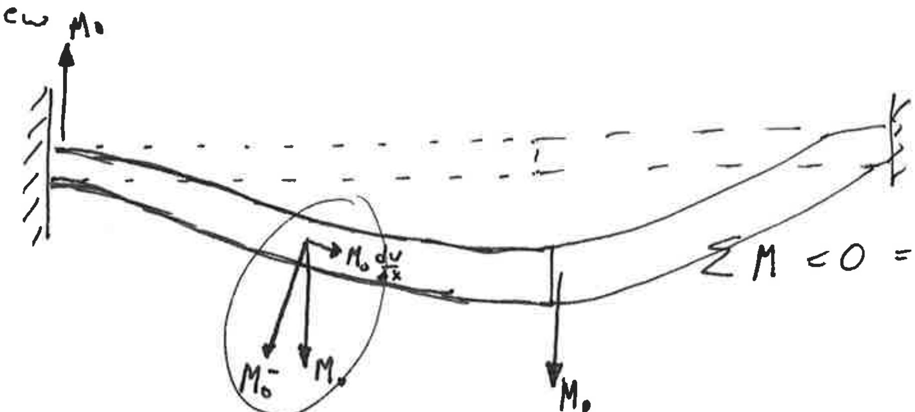
What if the loading causes the beam to twist?
Stable?
or move out of plane?

cross section

I_{yy} is lower and there is a new moment generated!

Pinned-Pinned Beam

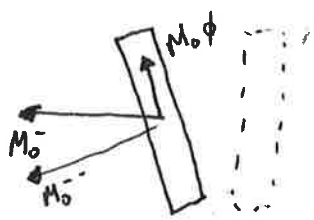
Top view



$$\Sigma M = 0 = M_{0_{LHS}} - M_{0_{RHS}}$$

But we can decompose the moment to moments along the beam's frame (since there is twisting and sideways deflection)

End View



Decompose $\tilde{M_0}$ into beam frame

Solving the bending and twisting equations

$$EI\,v'' = -M_o\,\phi \qquad\qquad \underbrace{\frac{Ght^3}{3}}_{C}\,\phi = M_o\,v'$$

gives a solution

$$\phi = \text{Constant}\cdot\underbrace{\sin\left(x\,\frac{M_o}{\sqrt{EI\cdot C}}\right)}_{\text{The critical case is when this is } \pi}$$

$$M_{crit} = \frac{\pi\sqrt{EI\cdot C}}{l}$$

Cantilever Beam

The math is much more complicated

Buckles at $P_{crit} = \dfrac{4.01}{l^2}\sqrt{G\frac{1}{3}ht^3\,E\,\frac{ht^3}{12}} \approx \dfrac{0.67\,ht^3}{l^2}\,\dfrac{E}{\sqrt{2(1+\nu)}}$

Failure?

Buckle or Yield?

P_{crit} $\quad$ $\sigma = \frac{My}{I}$

Ex: What is the buckling load for a 3 foot spar of height 1 in and thickness 0.1 in. The spar is Al.

$$P_{crit} = \frac{0.67 \cdot 1\,\cancel{in}\cdot 0.1^3\,\cancel{in^3}}{36^2\,\cancel{in^2}}\cdot\frac{10\times10^6\,\cancel{psi}}{\sqrt{2(1+.3)}}\cdot\frac{lbf}{\cancel{psi}\,\cancel{in^2}}$$

$= 3\ lbf$!

Ex. What is the max stress? $\quad M = 36\cdot F \quad I = \frac{1}{12}\cdot 0.1\cdot 1^3 = 0.00833$

$$\sigma = \frac{My}{I} = \frac{36\cdot\cancel{F}^{3}\cdot 0.5}{0.00833} = 6.4\ ksi$$

$\boxed{\text{Buckles}}$