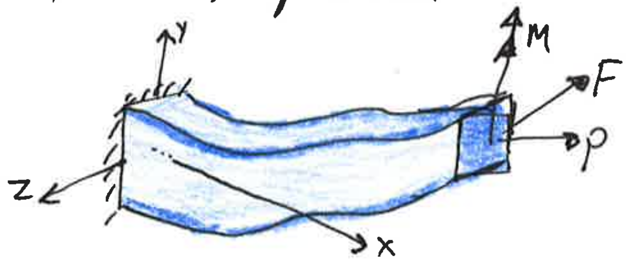
Advanced Beams

2 assumptions:

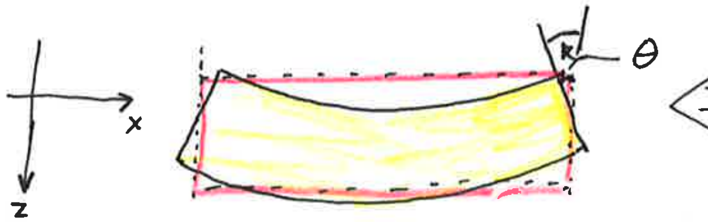
1) (σ_{yy} and σ_{zz}) < (σ_{xx})
2) $\underbrace{\text{planer cross sections}}_{\text{Euler - Bernoulli}}$

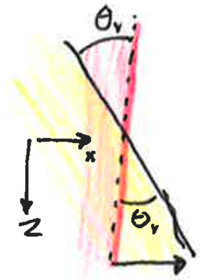
Aerospace Vehicles fit these!

Arbitrary loading of a beam



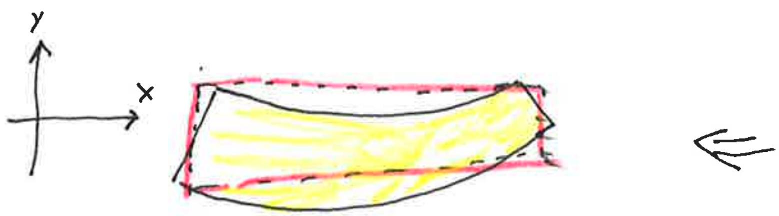
Take a small section and view from above

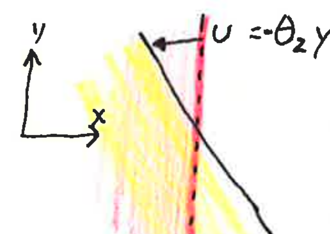




$\underbrace{U = \theta_y z}_{\text{since } d\ell = r\, d\theta}$

Now view from side





Notice the sign since $\overset{y}{\llcorner}_x$

What is the total deflection U at a point x, y, z?

$$U(x,y,z) = \underbrace{U_o}_{\text{axial extension}} + \underbrace{(\theta_y) z}_{\text{bending in z}} - \underbrace{(\theta_z) y}_{\text{bending in y}}$$

Take derivative wrt x

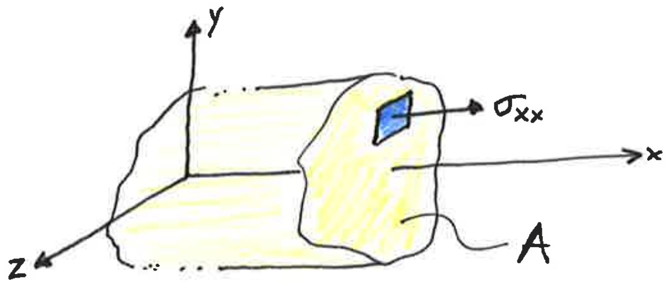
$$\frac{dU(x,y,z)}{dx} = \frac{dU_o}{dx} + \frac{d\theta_y}{dx} z - \frac{d\theta_z}{dx} y$$

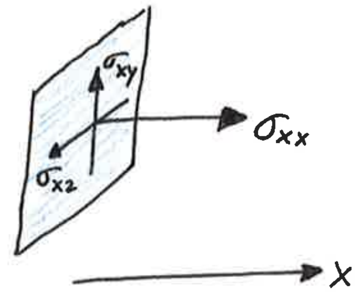
and obviously $\frac{dz}{dx} = 0 = \frac{dy}{dx}$ so there is no $\theta_y \frac{dz}{dx}$ term!

What is $\frac{dU}{dx}$? Strain! ε_{xx}

Stress $\boxed{\sigma_{xx} = E\varepsilon_{xx} = E\left(\frac{dU_o}{dx} + \frac{d\theta_y}{dx} z - \frac{d\theta_z}{dx} y\right)}$

Internal Resultant Forces and Moments

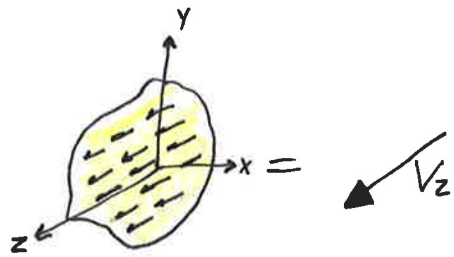




Axial force in x-dir "P" $= \int_A \sigma_{xx}\, dA$

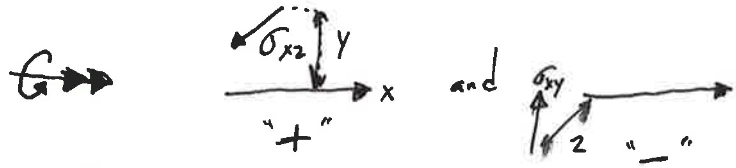
Shear

$$V_y = \int_A \sigma_{xy}\, dA \qquad V_z = \int_A \sigma_{xz}\, dA$$



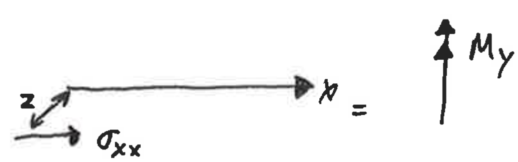
Moment "F·d" force times distance ≡ integral of stress times distance over an area.

M_x



$$M_x = \int_A (\sigma_{xz}\, y - \sigma_{xy}\, z)\, dA$$

$$M_y = \int_A \sigma_{xx}\, z\, dA$$



$$M_z = \int_A -\sigma_{xx}\, y\, dA$$

In other words, the internal forces and moments are just the integral of stresses. This doesn't solve for the stresses, but does limit/constrain them.

Back to our advanced Beam

We have an expression for stress $\sigma_{xx} = E\left(\frac{du_0}{dx} + \frac{d\theta_y}{dx}z - \frac{d\theta_z}{dx}y\right)$

- plug into the resultance force expressions

$$P = \int_A \sigma_{xx} dA = \int_A E\left(\frac{du_0}{dx} + \frac{d\theta_y}{dx}z - \frac{d\theta_z}{dx}y\right) dA$$

$$= \int_A E\frac{du_0}{dx} dA + \int_A E\frac{d\theta_y}{dx} z\, dA - \int_A E\frac{d\theta_z}{dx} y\, dA$$

And since our 2nd assumption was a planar cross section, the angle does not vary over dA!! pull it out!

$$P = E\frac{du_0}{dx}\underbrace{\int_A dA}_{A} + E\frac{d\theta_y}{dx}\underbrace{\int_A z\, dA}_{\bar{z}A} - E\frac{d\theta_z}{dx}\underbrace{\int y\, dA}_{\bar{y}A}$$

If we put the axis system on the centroid, then $\bar{z}$ and $\bar{y}$ are zero!

$$\boxed{P = EA\frac{du}{dx}}$$

- Plug into M_y resultant force equation

$$M_y = \int_A \sigma_{xx} z\, dA = \int_A E\frac{du_0}{dx} z\, dA + \int_A E\frac{d\theta_y}{dx} z^2 dA + \int_A -E\frac{d\theta_z}{dx} yz\, dA$$

$$= E\frac{du_0}{dx}\int_A z\, dA + E\frac{d\theta_y}{dx}\int_A z^2 dA - \int_A yz\, dA \left(E\frac{d\theta_z}{dx}\right)$$

$$\boxed{M_y = E\frac{du_0}{dx}\cancelto{0}{\bar{z}A} + E\frac{d\theta_y}{dx} I_{yy} - E\frac{d\theta_z}{dx} I_{yz}}$$

since $I_{yy} = \int_A z^2 dA$ and $I_{yz} = \int_A yz\, dA$

• $M_z = \int_A -\sigma_{xx}\, y\, dA$ = same process as above = $\boxed{E\frac{d\theta_z}{dx} I_{zz} - E\frac{d\theta_y}{dx} I_{yz} = M_z}$

So we have a system of equations

$$\boxed{\begin{aligned} P &= EA\frac{du_0}{dx} \\ M_y &= -E\frac{d\theta_z}{dx} I_{yz} + E\frac{d\theta_y}{dx} I_{yy} \\ M_z &= E\frac{d\theta_z}{dx} I_{zz} - E\frac{d\theta_y}{dx} I_{yz} \end{aligned}}$$

But really, we want to solve for $\frac{du_0}{dx}$, $\frac{d\theta_y}{dx}$ and $\frac{d\theta_z}{dx}$

$$\begin{pmatrix} P \\ M_y \\ M_z \end{pmatrix} = \begin{bmatrix} EA & & \\ & EI_{yy} & -EI_{yz} \\ & -EI_{yz} & EI_{zz} \end{bmatrix} \begin{pmatrix} \frac{du_0}{dx} \\ \frac{d\theta_y}{dx} \\ \frac{d\theta_z}{dx} \end{pmatrix}$$

This is a block diagonal 1x1 + 2x2. Invert 3x3 to give

$$\boxed{\frac{du_0}{dx} = \frac{P}{AE}}$$

$$\begin{pmatrix} \frac{d\theta_y}{dx} \\ \frac{d\theta_z}{dx} \end{pmatrix} = \frac{\begin{bmatrix} EI_{zz} & EI_{yz} \\ EI_{yz} & EI_{yy} \end{bmatrix}}{E^2 I_{yy} I_{zz} - E^2 I_{yz} I_{yz}} \begin{pmatrix} M_y \\ M_z \end{pmatrix}$$

$$\boxed{= \frac{1}{E} \cdot \frac{1}{I_{yy}I_{zz} - I_{yz}^2} \cdot \begin{bmatrix} I_{zz} & I_{yz} \\ I_{yz} & I_{yy} \end{bmatrix} \begin{pmatrix} M_y \\ M_z \end{pmatrix}}$$

So the total strain is

$$\varepsilon_{xx} = \frac{du}{dx} = \frac{du_0}{dx} - y\frac{d\theta_z}{dx} + z\frac{d\theta_y}{dx}$$ substitute

$$\boxed{\varepsilon_{xx} = \frac{P}{AE} - \frac{1}{E}\frac{I_{yz}M_y + I_{yy}M_z}{I_{yy}I_{zz} - I_{yz}^2}\, y + \frac{1}{E}\frac{I_{zz}M_y + I_{yz}M_z}{I_{yy}I_{zz} - I_{yz}^2}\, z}$$

The total stress is $\sigma_{xx} = E\,\varepsilon_{xx}$

$$\boxed{\sigma_{xx} = \frac{P}{A} - \frac{I_{yz} M_y + I_{yy} M_z}{I_{yy} I_{zz} - I_{yz}^2}\, y + \frac{I_{zz} M_y + I_{yz} M_z}{I_{yy} I_{zz} - I_{yz}^2}\, z}$$

Remember previously that Inertias could be transformed such that $I_{yz} = 0$
(similar to transforming stresses with Mohr's Circle)

If $I_{yz} = 0$

$$\sigma_{xx} = \frac{P}{A} - \frac{I_{yy} M_z}{I_{yy} I_{zz}}\, y + \frac{I_{zz} M_y}{I_{yy} I_{zz}}\, z$$

$$\boxed{\sigma_{xx} = \frac{P}{A} - \underbrace{\frac{M_z y}{I_{zz}}}_{\frac{My}{I}} + \frac{M_y}{I_{yy}}\, z}$$

you recognize this from basic str' of mat'

If no bending moments

$$\boxed{\sigma_{xx} = \frac{P}{A}}$$

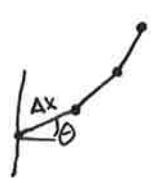
Q: Can you find the deflection from this?

A: yes!

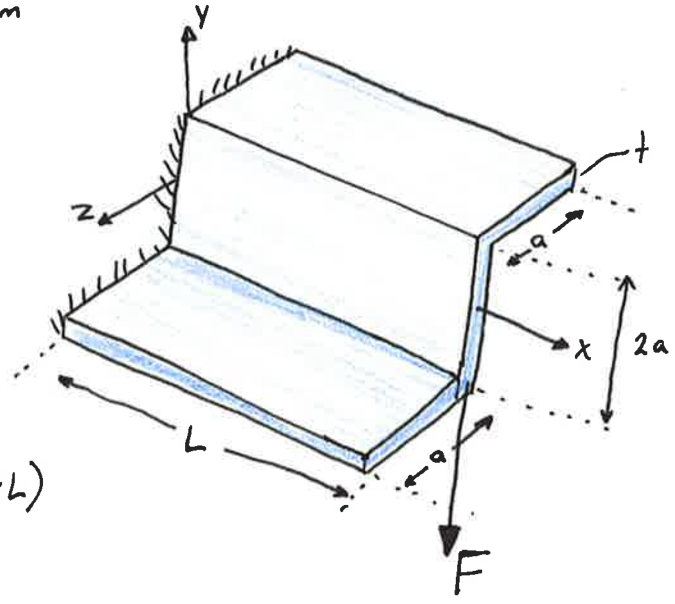
$$\frac{du_0}{dx} = \frac{P}{AE} \Rightarrow \int_0^{u(x)} du_0 = \int_0^x \frac{P}{AE}\, dx \Rightarrow u(x) = \frac{P}{AE}\, x$$

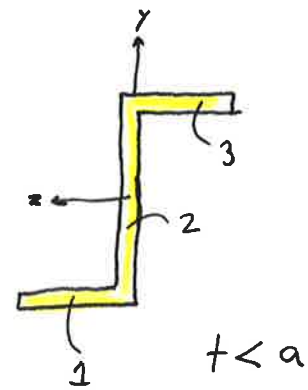
A: Also for angles!

$$\frac{d\theta_y}{dx} = \frac{1}{E} \frac{I_{zz} M_y + I_{yz} M_z}{I_{yy} I_{zz} - I_{yz}^2} \Rightarrow \int d\theta_y = \int (\cdots)\, dx = \theta_y$$



Ex: Z beam





① $M_z = F\cdot(x-L)$

② MoI

$$I_{yy} = \underbrace{\tfrac{1}{12} t a^3 + t a \left(\tfrac{a}{2}\right)^2}_{\#1} + \underbrace{\tfrac{1}{12}(2a)t^3}_{\#2} + \underbrace{\tfrac{1}{12} t a^3 + t a\left(\tfrac{a}{2}\right)^2}_{\#3} = \tfrac{1}{6} t a^3 + \tfrac{1}{2} t a^3 + \overset{\approx 0}{\tfrac{1}{6} a t^3}$$

$$\approx \tfrac{2}{3} t a^3$$

$$I_{zz} = \underbrace{\tfrac{1}{12} a \overset{0}{\cancel{t^3}} + at\cdot a^2}_{\#1} + \underbrace{\tfrac{1}{12} t (2a)^3}_{\#2} + \underbrace{\text{same as \#1}}_{\#3} \approx 2a^3 t + \tfrac{8}{12} t a^3$$

$$\approx \left(2 + \tfrac{2}{3}\right) a^3 t = \tfrac{8}{3} a^3 t$$

$$I_{yz} = \overset{0 \text{ for sym part}}{\cancel{I^{i}_{y_0 z_0}}} + \bar{y}\bar{z}A = \underbrace{\underbrace{(-a)}_{\bar{y}}\underbrace{\left(\tfrac{a}{2}\right)}_{\bar{z}}\underbrace{(at)}_{A}}_{\#1} + \underbrace{0\cdot 0\cdot 2at}_{\#2} + \underbrace{(a)\left(-\tfrac{a}{2}\right)(at)}_{\#3}$$

$$= -a^3 t$$

Could we transform the coordinate system such that $I_{yz} = 0$?

Yes, but the Force F would also need to be transformed, yielding M_z and M_y.

Compute stresses

$$\sigma_{xx} = \cancel{\frac{P}{A}}^{0} - \frac{M_z I_{yy} + \cancel{M_y}^{0} I_{yz}}{I_{yy} I_{zz} - I_{yz}^2}\, y + \frac{\cancel{M_y}^{0} I_{zz} + M_z I_{yz}}{I_{yy} I_{zz} - I_{yz}^2}\, z$$

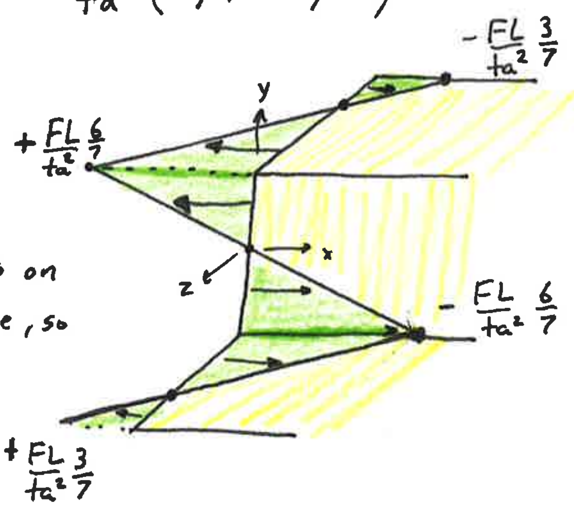
$$= - \frac{F(x-L)\,\frac{2}{3} t a^3}{\frac{2}{3} t a^3\, \frac{8}{3} a^3 t - a^6 t^2}\, y + \frac{F(x-L)\,(-a^3 t)}{\frac{2}{3} t a^3\, \frac{8}{3} a^3 t - a^6 t^2}\, z$$

$$= - F(x-L)\, \frac{6}{7}\, \frac{1}{t a^3}\, y \quad - F(x-L)\, \frac{9}{7}\, \frac{1}{t a^3}\, z$$

$$= \frac{-F(x-L)}{7 t a^3}\left(6y + 9z\right)$$

At $x = 0$,

$$\sigma_{xx} = \frac{FL}{t a^3}\left(\frac{6}{7} y + \frac{9}{7} z\right)$$



Notice this is on the back face, so + is ←

at origin
$\sigma_{xx} = 0$

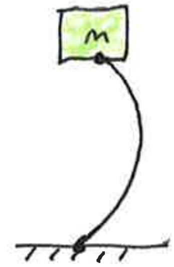
at bends
$(a, 0) \Rightarrow \sigma_{xx} = + \frac{FL}{ta^2}\frac{6}{7}$

at edge of flange
$(a, -a) \Rightarrow \pm \frac{FL}{ta^2}\frac{3}{7}$

zero σ_{xx}
at $(a, -\frac{2}{3}a)$
and $(-a, \frac{2}{3}a)$

Euler ~~Beam~~ Column Buckle

How heavy can the mass be before the beam buckles



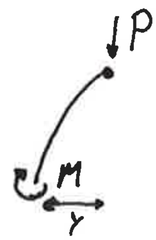
Boundary Conditions?
Differs for pinned vs fixed

Theory

pinned - pinned beam (i.e free to rotate at ends, but only x deflection)

What is the moment at any point?

$$M = -Py$$



Beam equation for bending

$$EIy'' = M = -Py \Rightarrow \underline{EIy'' + Py = 0}$$

Solve this ODE

$$y = A\sin\left(\sqrt{\frac{P}{EI}}\,x\right) + B\cos\left(\sqrt{\frac{P}{EI}}\,x\right)$$

Apply BCs.

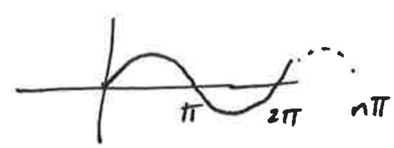
$y(0) = 0$ removes B term since $\cos(0) \neq 0$

$$y = A\sin\left(\sqrt{\frac{P}{EI}}\,x\right)$$

Apply other BC.

$$y(L) = 0 = A\sin\left(\sqrt{\frac{P}{EI}}\,L\right)$$

either $A = 0$ <u>or</u> $\sin\left(\sqrt{\frac{P}{EI}}\,L\right) = 0$



Thus, $\sqrt{\frac{P}{EI}}\,L = \pi n$

Solve for P

$$P = \frac{\pi^2 n^2 EI}{L^2}$$

lowest P when $n = 1$

$$\boxed{P_{crit} = \frac{\pi^2 EI}{L^2}}$$

Ex: Al pushrod for FCS.

10 ft Al pushrod of 1.5 in diameter, 1/8 in thickness
Find max force

$$P = \frac{\pi^2 \;\Big|\; 10\times10^6 \frac{lbf}{in^2} \;\Big|\; \pi \;\Big|\; 0.75^3\, in^3 \;\Big|\; 0.125\, in \;\Big|\; ft^2}{10^2\, ft^2 \;\Big|\; 144\, in^2} = \boxed{1130\ lbf}$$