Heterogeneous Beams with Thermal loading

Remember from earlier, we found an expression for the strain from geometry.

$$\varepsilon_{xx} = \frac{du_0}{dx} - y\frac{d\theta_z}{dx} + z\frac{d\theta_y}{dx}$$

And earlier in the materials portion of the lectures found that temperature creates a strain

$$\varepsilon_{xx} = \alpha \Delta T$$

So, we could combine to give the stress (assuming σ_{yy} and σ_{zz} are $\ll \sigma_{xx}$)

$$\sigma_{xx} = E\frac{du_0}{dx} - Ey\frac{d\theta_z}{dx} + Ez\frac{d\theta_y}{dx} - \underbrace{E\alpha\Delta T}_{\text{C.f chap 3 planar and 3D cases. This assumes 1D case!}}$$

Plugging into the resultant Force & Moment Equations

$$P = \int_A E\left(\frac{du_0}{dx} - y\frac{d\theta_z}{dx} + z\frac{d\theta_y}{dx} - \alpha\Delta T\right) dA$$

$$M_y = \int_A E\left(\frac{du_0}{dx} - y\frac{d\theta_z}{dx} + z\frac{d\theta_y}{dx} - \alpha\Delta T\right) z\, dA$$

$$M_z = \int_A E\left(\cdots \cdots + \alpha\Delta T\right) y\, dA$$

Exactly the same as before but with $-\alpha\Delta T$ temperature term.

In actual aerospace vehicles, different materials are common in structural parts. (e.g. Fiberglass + Al, Steel + Al, Al + plastics)

$\boxed{\text{E is no longer constant!!}}$

Standard practise is to pick a reference value for E (e.g. 1×10^6 psi or 1 psi or ......

$$P = \underbrace{\left(E_1\int_A \frac{E}{E_1} dA\right)}_{A^*}\frac{du_0}{dx} - \underbrace{\left(E_1\int_A \frac{E}{E_1} y\, dA\right)}_{\bar{y}^*}\frac{d\theta_z}{dx} + \underbrace{\left(E_1\int_A \frac{E}{E_1} z\, dA\right)}_{\bar{z}^*}\frac{d\theta_y}{dx} - \underbrace{\int_A E\alpha\Delta T\, dA}_{P^T}$$

$$\boxed{P = E_1 A_1^* \frac{du_0}{dx} - E_1 \bar{y}^* \frac{d\theta_z}{dx} + E_1 \bar{z}^* \frac{d\theta_y}{dx} - P^T}$$

Doing the same process to M_y and M_z gives (after rearrangin)

$$P + P^T = E_1 A^* \frac{du_0}{dx}$$

$$M_y + M_y^T = -E_1 I_{yz}^* \frac{d\theta_z}{dx} + E_1 I_{yy}^* \frac{d\theta_y}{dx}$$

$$M_z - M_z^T = E_1 I_{zz}^* \frac{d\theta_z}{dx} - E_1 I_{yz}^* \frac{d\theta_y}{dx}$$

where

Centroid → $A^* = \int_A \frac{E}{E_1} dA$ $\quad \bar{y}^* = \frac{1}{A^*}\int_A \frac{E}{E_1} y\, dA$ $\quad \bar{z}^* = \frac{1}{A^*}\int_A \frac{E}{E_1} z\, dA$

MoI → $I_{yy}^* = \int_A \frac{E}{E_1} z^2 dA$ $\quad I_{yz}^* = \int_A \frac{E}{E_1} yz\, dA$ $\quad I_{zz}^* = \int \frac{E}{E_1} y^2 dA$

Thermal loads → $P^T = \int_A E \alpha \Delta T\, dA$ $\quad M_y^T = \int_A E \alpha \Delta T z\, dA$ $\quad M_z^T = \int_A E \alpha \Delta T y\, dA$

As before, solve for $\frac{du_0}{dx}$, $\frac{d\theta_z}{dx}$, and $\frac{d\theta_y}{dx}$ with a 3x3 inverse

$$\frac{du_0}{dx} = \frac{P + P^T}{E_1 A^*}$$

$$\frac{d\theta_z}{dx} = \frac{(M_z - M_z^T) I_{yy}^* + (M_y + M_y^T) I_{yz}^*}{E_1 (I_{yy}^* I_{zz}^* - I_{yz}^{*2})}$$

$$\frac{d\theta_y}{dx} = \frac{(M_y + M_y^T) I_{zz}^* + (M_z - M_z^T) I_{yz}^*}{E_1 (I_{yy}^* I_{zz}^* - I_{yz}^{*2})}$$

Plugging back into the $\varepsilon_{xx} = \frac{du_0}{dx} - y\frac{d\theta_z}{dx} + z\frac{d\theta_y}{dx}$ equation gives

$$\boxed{\begin{aligned}\varepsilon_{xx} = \frac{P+P^T}{E_1 A^*} &- \frac{1}{E_1}\left(\frac{(M_z - M_z^T) I_{yy}^* + (M_y + M_y^T) I_{yz}^*}{I_{yy}^* I_{zz}^* - I_{yz}^{*2}}\right) y \\ &+ \frac{1}{E_1}\left(\frac{(M_y + M_y^T) I_{zz}^* + (M_z - M_z^T) I_{yz}^*}{I_{yy}^* I_{zz}^* - I_{yz}^{*2}}\right) z\end{aligned}}$$

and stress $\sigma = E\varepsilon - E\alpha\Delta T$

$$\boxed{\begin{aligned}\sigma_{xx} = \frac{E(P+P^T)}{E_1 A^*} &- \frac{E}{E_1}\left(\frac{(M_z - M_z^T) I_{yy}^* + (M_y + M_y^T) I_{yz}^*}{I_{yy}^* I_{zz}^* - I_{yz}^{*2}}\right) y \\ &+ \frac{E}{E_1}\left(\frac{(M_y + M_y^T) I_{zz}^* + (M_z - M_z^T) I_{yz}^*}{I_{yy}^* I_{zz}^* - I_{yz}^{*2}}\right) z \\ &- E\alpha\Delta T\end{aligned}}$$

The stress in a composite heterogeneous advanced beam with thermal loads depends on:

- Material E
- Coefficient of expansion
- Temperature
- Geometry

Ex:

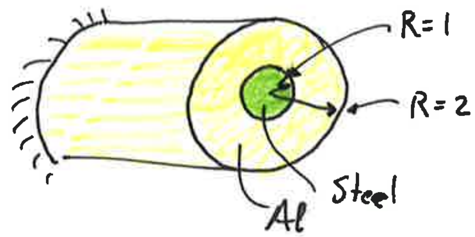
Ignore bending and investigate the temperature vs. stress in a uniform bar

$$\sigma_{xx} = \frac{E}{E_1 A^*}(P+P^T) - E\alpha\Delta T = \cancel{\frac{E}{E_1}}\frac{1}{\int \cancel{\frac{E}{E_1}} dA}\left(P + \int E\alpha\Delta T\, dA\right) - E\alpha\Delta T$$

$$= \left(\frac{P}{A} + \frac{E\alpha\Delta T A}{A}\right) - E\alpha\Delta T = \frac{P}{A}$$

recovers the basic stress equation

Simple Symmetrical Example



Steel $\quad E = 30\times10^6$ psi $\quad \alpha = 7\times10^{-6}$

Al $\quad E = 10\times10^6$ psi $\quad \alpha = 13\times10^{-6}$

pick $E_1 = E = 10\times10^6$ psi

- No loading (i.e. no bending and no axial force and no temp)

$\sigma_{xx} = 0$ by inspection

- $\Delta T = 100^F$

$$\sigma_{xx} = \frac{E}{E_1 A^*}(\cancelto{0}{P} + P^T) - E\alpha\Delta T = \frac{E P^T}{E_1 A^*} - E\alpha\Delta T$$

$$A^* \equiv \int_A \frac{E}{E_1}\,dA = \sum_i \frac{E_i}{E_1} A_i = \underbrace{\cancelto{1}{\frac{10\times10^6}{10\times10^6}}\left(\frac{\pi}{4}\right)\left(\cancelto{16-4=12}{4^2-2^2}\right)}_{\text{Al outer}} + \underbrace{\cancelto{3}{\frac{30}{10}}\left(\frac{\pi}{4}\right)\left(\cancelto{4}{2^2}\right)}_{\text{Steel inner}}$$

$$= 3\pi + 3\pi = 6\pi \approx 18.849 \text{ in}^2$$

$\bar{z}^* = \bar{y}^* = 0$ by inspection

$$P^T = \int_A E\alpha\Delta T\,dA = \sum E_i\alpha_i\Delta T_i A_i = 10\times10^6\cdot13\times10^{-6}\cdot100\cdot\frac{\pi}{4}(4^2-2^2)$$
$$+\ 30\times10^6\cdot7\times10^{-6}\cdot100\cdot\frac{\pi}{4}(2^2)$$

$$= 188495.6 \text{ lbf}$$

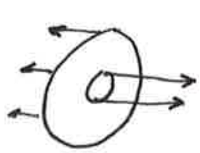
In the Al,

$$\sigma_{xx} = \cancelto{1}{\frac{E}{E_1}}\frac{P^T}{A^*} - E_{Al}\alpha\Delta T = \frac{188495.6}{18.849} - 10\times10^6\cdot13\times10^{-6}\cdot100$$

$$= 10000 - 13000 = \boxed{-3000 \text{ psi}} \text{ Al}$$

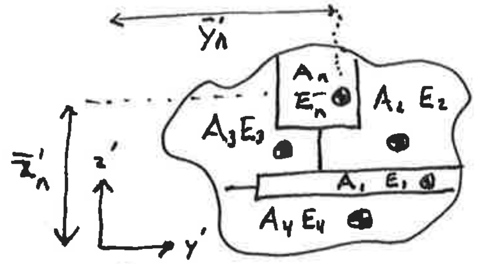
In the steel

$$\sigma_{xx} = 3\cdot\frac{188495.6}{18.849} - 30\times10^6\cdot7\times10^{-6}\cdot100 = \boxed{9000 \text{ psi}} \text{ Steel}$$



$$\int\sigma_{xx}\,dA = 0 = (9\times10^3 \text{ psi})(1) - (3000)(3) = 0 \checkmark$$

Modulus Weighted Section Properties



From our derivation,

$$A^* = \int_A \frac{E}{E_1}\,dA = \sum_i \int_A \frac{E_i}{E_1}\,dA = \sum \frac{E_i}{E_1}\int_A dA = \underline{\sum \frac{E_i}{E_1} A_i}$$

$$\bar{y}^* = (\ldots) = \frac{1}{A^*}\sum_i \frac{E_i}{E_1}\bar{y}_i A_i$$

$$\bar{z}^* = \frac{1}{A^*}\sum_i \frac{E_i}{E_1}\bar{z}_i A_i$$

Moments of Inertia can be tricky (or confusing) (E_i varies and $\bar{y}^*$ and $\bar{y}'$ and y and ....)

① Compute modulus weighted $I^*_{y'y'}$ about a convient axis system $y'\ z'$

$$I^*_{y'y'} = \sum_i \frac{E_i}{E_1}\Big(\underbrace{I_{y_0y_0\,i}}_{\text{"regular" MoI}} + \underbrace{\bar{z}_i'^{\,2} A_i}_{\text{distance}^2\text{ from axis system } y'z' \text{ and actual area}}\Big)$$

② Shift back to modulus weighted centroid

$$I^*_{yy} = \underbrace{I^*_{y'y'}}_{\text{above}} - \underbrace{(\bar{z}'^*)}_{\text{modulus weighted Centroid}}{}^2 \underbrace{A^*}_{\text{modulus weighted area of entire cross section}}$$

Similarly for I^*_{yz} and I^*_{zz}

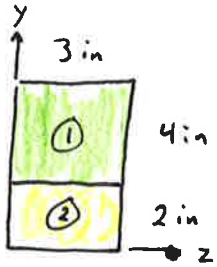
$$I^*_{y'z'} = \sum_i \frac{E_i}{E_1}\left(I_{y_0z_0\,i} + \bar{y}_i'\bar{z}_i' A_i\right)$$

$$I^*_{z'z'} = \sum_i \frac{E_i}{E_1}\left(I_{z_0z_0\,i} + \bar{y}_i'^{\,2} A_i\right)$$

$$I^*_{yz} = I^*_{y'z'} - \bar{y}'^*\bar{z}'^* A^*$$

$$I^*_{zz} = I^*_{z'z'} - \bar{y}'^{*2} A^*$$

Ex. 4.6



① $E = 30\times10^6$ psi $\alpha = 13\times10^{-6}$

② $E = 10\times10^6$ psi $\alpha = 6.5\times10^{-6}$

↑ E_1 reference

} Notice that these are backwards for Al and Steel!

① Centroid

$$A^* = \sum \frac{E_i}{E_1} A_i = 3\cdot12 + 1\cdot6 = 36+6 = 42$$

$$\bar{y}^* = \frac{1}{A^*}\sum \frac{E_i}{E_1} y A_i = 3\cdot4\cdot12 + 1\cdot1\cdot6 = 144+6 = 150/42 = 3.57 \text{ in}$$

$$\bar{z}^* = \frac{1}{A^*}\sum \frac{E_i}{E_1} z A_i = 3\cdot1.5\cdot12 + 1\cdot1.5\cdot6 = 63/42 = 1.5 \text{ in}$$

② Thermal Loads (if $\Delta T = 100$)

$$P^T = \sum E_i \alpha \Delta T A_i = 30\times10^6\cdot13\times10^{-6}\cdot100\cdot12 + 10\times10^6\cdot6.5\times10^{-6}\cdot100\cdot6$$

$$= 507000 \text{ lbf}$$

$$M_y^T = \sum E_i \alpha_i \Delta T_i z A_i = \cancel{30\times10^6\ 13\times10^{-6}\ 100}$$ let's put this in a tabular format

part	A_i	E_i	α_i	ΔT	P^T: $E_i\alpha\Delta T A_i$	$\bar{z}=z-\bar{z}^*$	$\underbrace{M_y}_{E_i\alpha\Delta T z A_i}$	$\bar{y}=y-\bar{y}^*$	$\underbrace{M_z^T}_{E_i\alpha_i\Delta T y A_i}$
1	12	30×10^6	13×10^{-6}	100	468×10^3	1.5 − 1.5	0	1 − 3.57	-1.2×10^6
2	6	10×10^6	6.5×10^{-6}	100	39×10^3	1.5 − 1.5	0	4 − 3.57	38999.6
					507×10^3		0		-1.16×10^6

Actually, ΔT in problem = 0 so $M_y^T = 0$, $M_z^T = 0$, $P^T = 0$

③ Moments of Inertia

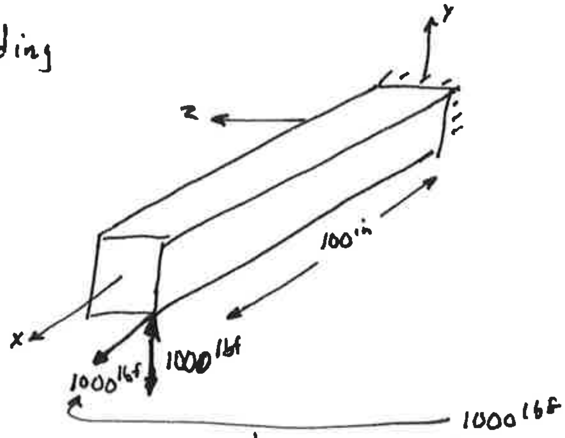
Part	A_i	E_i	$\bar{y}'$	$\bar{z}'$	$I_{y_0y_0}$	$I_{z_0z_0}$	$\frac{E_i}{E_1}\left(I_{y_0y_0}+\bar{z}^2A_i\right)$	$\frac{E_i}{E_1}\left(I_{z_0z_0}+\bar{y}^2A_i\right)$
1	12	30×10^6	4	1.5	9	16	108	624
2	6	10×10^6	1	1.5	4.5	2	18	8
							126	632

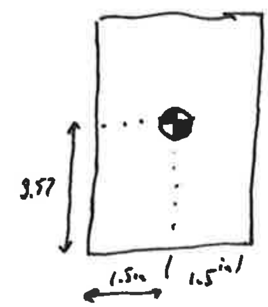
shift to M.W Centroid

$$I_{yy}^* = 126 - (1.5)^2\cdot42 = 31.5 \text{ in}^4$$

$$I_{zz}^* = 632 - (3.57)^2\,42 = 96.7 \text{ in}^4$$

④ Loading





$$\frac{dp}{dx} = -\cancel{p_x(x)}^{0} \overset{\text{integrate}}{\Rightarrow} p(x) - \cancel{p(L)} = 0 \quad \Rightarrow \quad p(x) = 1000^{\text{lbf}}$$

$$\frac{dV_y}{dx} = -\cancel{p_y(x)}^{0} \overset{\text{integrate}}{\Rightarrow} V_y(x) - \cancel{V_y(L)}^{1000} = 0 \quad \Rightarrow \quad V_y(x) = 1000^{\text{lbf}}$$

No V_z loading $V_z = 0$

$$\frac{dM_x}{dx} = -\cancel{m_x(x)}^{0} \Rightarrow M_x(x) - \cancel{M_x(L)}^{1000^{\text{lb}} \cdot 1.5^{\text{in}}} = 0 \Rightarrow M_x(x) = 1500^{\text{lbf-in}}$$

$$\frac{dM_y}{dx} = -\cancel{m_y(x)}^{0} + \cancel{V_z(x)}^{0} = 0 \Rightarrow M_y - \cancel{M_y(L)}^{-1.5\text{in} \cdot 1000^{\text{lbf}}} = 0 \qquad M_y(x) = -1500^{\text{lbf-in}}$$

$$\frac{dM_z}{dx} = -\cancel{m_z(x)}^{0} - \cancel{V_y(x)}^{\substack{\text{from above} \\ 1000}} \Rightarrow$$

write out formally to not make mistakes!

$$dM_z = -1000\,dx \qquad \text{integrate} \quad \int_{M_z(L)}^{M_z(x)} dM_z = \int_L^x -1000\,dx$$

$$M_z(x) - \cancel{M_z(L)}^{1000^{\text{lbf}} \cdot 3.57^{\text{in}}} = -1000x\Big|_L^x = -1000x + 1000\cancel{L}^{100\text{ in}}$$

$$M_z(x) = -1000x + 100000 + 3570$$

$$M_z(x) = -1000x + 103570 \ \text{lbf-in}$$

⑤ Stresses

$$\sigma = \frac{E}{E_1 A^*}\left(P + \cancel{P^T}\right) - \frac{E}{E_1}\left(\frac{(M_z - \cancel{M_z^T}) I_{yy}^* + (M_y + M_y^T)\cancel{I_{yz}^*}}{I_{yy}^* I_{zz}^* - \cancel{I_{yz}^{*2}}}\right) y$$

$$+ \frac{E}{E_1}\left(\frac{(M_y - \cancel{M_y^T}) I_{zz}^* + (M_z - M_z^T)\cancel{I_{yz}^*}}{I_{yy}^* I_{zz}^* - \cancel{I_{yz}^{*2}}}\right) z - E\alpha\cancel{\Delta T}$$

$$= \frac{E P}{E_1 A^*} - \frac{E}{E_1}\frac{M_z}{I_{zz}^*} y + \frac{E}{E_1}\frac{M_y}{I_{yy}^*} z$$

top part ①

$$\boxed{\sigma = \frac{3 \cdot 1000^{lbf}}{42^{in^2}} - \frac{3(-1000x + 103570)}{96.7} y + \frac{3(-1500)}{31.5} z}$$

bottom part ②

$$\boxed{\sigma = \frac{1000^{lbf}}{42^{in^2}} - \frac{(-1000x + 103570)}{96.7} y + \frac{(-1500)}{31.5} z}$$

$$\sigma_{xx}(\text{top}) = \sigma_{xx}(0, 2.43, 0) = -7.7^{ksi}$$

$$\sigma_{xx}(\text{bottom}) = \sigma_{xx}(0, -3.57, 0) = +3.8^{ksi}$$