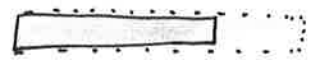
Deformation (Bending + Extension) NO TWISTING

- Axial Extension

$$\varepsilon_{xx} = \frac{du_o}{dx} = \frac{P + P^T}{E_1 A^*}$$

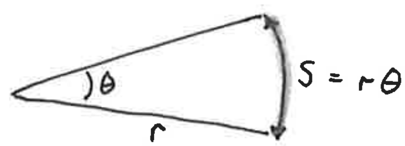


Integrate $\frac{du_o}{dx}$

$$\int_{u_o(0)}^{u_o(x)} du_o = \int_0^x \left(\frac{P+P^T}{E_1 A^*}\right) dx \quad \Rightarrow \quad u_o(x) = \int_0^x \frac{P+P^T}{E_1 A^*}\, dx$$

- Transverse Bending

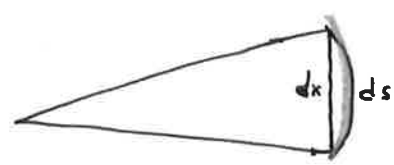
Remember from geometry, arc length = angle · radius



The differential is $dS = r\, d\theta$

or $\frac{d\theta}{dS} = \frac{1}{r}$

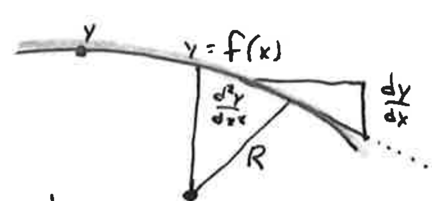
For small angles, $ds \approx dx$



$$\frac{d\theta}{dx} = \frac{1}{r}$$

The radius of curvature for a function $y=f(x)$

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$



Notice that $\left(\frac{dy}{dx}\right)^2$ for small deflection gives $R = \frac{1}{\left(\frac{d^2y}{dx^2}\right)}$

And, we have $\frac{d\theta}{dx}$ from a previous lesson

See 4.53 in book

Deflection in y

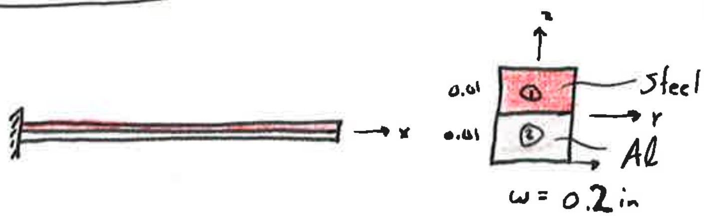
$$\boxed{\frac{d^2 V_0}{dx^2} = \frac{\left(M_z - M_z^T\right) I_{yy}^* + \left(M_y + M_y^T\right) I_{yz}^*}{E_1 \left(I_{yy}^* I_{zz}^* - I_{yz}^{*2}\right)}}$$

Deflection in z

$$\boxed{\frac{d^2 W_0}{dx^2} = \frac{-\left(M_y + M_y^T\right) I_{zz}^* - \left(M_z - M_z^T\right) I_{yz}^*}{E_1 \left(I_{yy}^* I_{zz}^* - I_{yz}^{*2}\right)}}$$

Ex:

Bimetallic Strip A useful and common application of thermal strain.



$h = 0.020$ in total

part	E_i/E_1	A	$E/E\,A$	$\bar{z}'$	$E/E\,\bar{z}A$	$I_{y_0y_0}$	$\frac{E}{E}\left(I_{y_0y_0} + \bar{z}^2 A\right)$
1	3	0.002	0.006	0.005	3.0×10^{-5}	1.66×10^{-9}	2.0×10^{-7}
2	1	0.002	0.002	-0.005	-1×10^{-5}	1.66×10^{-8}	6.6×10^{-8}
			0.008		2.0×10^{-5}		2.66×10^{-7}

$\bar{z}^* = 0.0025$ in

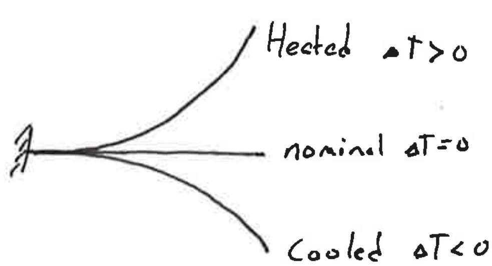
$$I_{yy}^* = I_{y_0y_0}^* - \bar{z}^* A^* = 2.66\times10^{-7} - 0.0025^2 \cdot 0.008$$
$$= 2.16\times10^{-7}$$

$P^T = \int E\alpha\,\Delta T\,dA$ = ignore for now

$$M_y^T = \sum E_i\,\alpha_i\,z_i\,A_i\,\Delta T$$
$$= 30\times10^6 \cdot 6.5\times10^{-6} \cdot 0.0075 \cdot (0.2 \cdot 0.01)\,\Delta T$$
$$+ 10\times10^6 \cdot 13\times10^{-6} \cdot -0.0125 \cdot (0.2 \cdot 0.01)\,\Delta T$$
$$= -0.000325\,\Delta T$$

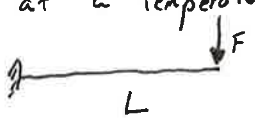
$$\frac{d^2 w_0}{dx^2} = \frac{-M_y^T \cancel{I_{zz}^*}}{E_1 I_{yy}^* \cancel{I_{zz}^*}} = \frac{+0.000325\,\Delta T}{10\times10^6 \cdot 2.16\times10^{-7}} = +0.00015\,\Delta T$$

$$W_0 = \iint 0.00015\,\Delta T\,dx\,dx = \frac{0.00015\,\Delta T\,x^2}{2}$$



Turn on/off a switch at a temperature.

$M_y + M_y^T = 0 \Rightarrow$



4.17 deflection

① Moments (from previous notes) (where I completely neglected the 1000 lbf force in y!!)

$$M_z = 0 \qquad M_y = -25x^2 + 5000x - 25000$$

② Moments of inertia

$$I_{yy}^* = I_{zz}^* = 70.95 \text{ in}^4 \qquad I_{yz}^* = -12.8 \text{ in}^4$$

③ deflection

$$\frac{d^2v}{dx^2} = \frac{(\overset{0}{\cancel{M_z}} - \overset{0}{\cancel{M_z^T}}) I_{yy}^* + (M_y + \overset{0}{\cancel{M_y^T}}) I_{yz}^*}{E_1 (I_{yy}^* I_{zz}^* - I_{yz}^{*2})}$$

$$= \frac{M_y I_{yz}^*}{E_1 (I_{yy}^* I_{zz}^* - I_{yz}^{*2})} = \frac{\overset{\text{lbf in}}{(-25x^2 + 5000x - 25000)}(-12.8 \text{ in}^4)}{10 \times 10^6 \frac{\text{lbf}}{\text{in}^2} (70.95^2_{\text{in}^8} - 12.8^2)} \qquad [\tfrac{1}{\text{in}}]$$

$$= \underbrace{-2.6 \times 10^{-10}}_{c} (-25x^2 + 5000x - 25000)$$

Integrate twice to give V BCs are $V(0) = 0$ (no motion) $V'(0) = 0$ (no slope)

$$\int_{V(0)=0}^{V(x)} \int_{\frac{dV}{dx}(0)=0}^{V(x)} d^2V = \int_0^x \int_0^x c(-25x^2 + 5000x - 25000)\, dx\, dx$$

$$dV(x) = c\left(-\frac{25x^3}{3} + \frac{5000x^2}{2} - 25000x\right) dx$$

$$V(x) = c\left(-\frac{25x^4}{12} + \frac{5000x^3}{6} - \frac{25000x^2}{2}\right)$$

$$V(100) = c\left(-\frac{25 \cdot 100^4}{12} + \frac{5000 \cdot 100^3}{6} - \frac{25000 \cdot 100^2}{2}\right)$$

$$\boxed{V(100) = -0.13 \text{ in}}$$

(4) $$\frac{d^2w}{dx^2} = \frac{-(M_y + M_y^T) I_{zz}^* - (M_z - M_z^T) I_{yz}^*}{E_1 (I_{yy}^* I_{zz}^* - I_{yz}^{*2})}$$

$$= \frac{-(-25x^2 + 5000x - 25000) I_{zz}^*}{E_1 (I_{yy}^* I_{zz}^* - I_{yz}^{*2})}$$

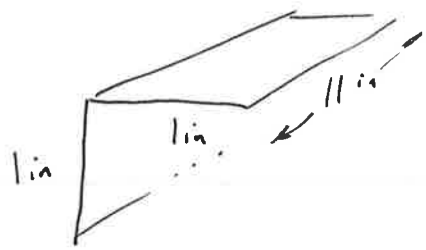
Similar integral as above except constant is different

$$C = -\frac{I_{zz}^*}{E_1 (I_{yy}^* I_{zz}^* - I_{yz}^{*2})} = -1.46 \times 10^{-9}$$

$$w(x) = -1.146 \times 10^{-9} \left(\frac{-25\, x^4}{12} + \frac{5000}{6} x^3 - \frac{25000}{2} x^2 \right)$$

$$\boxed{w(100) = -0.72 \text{ in}}$$

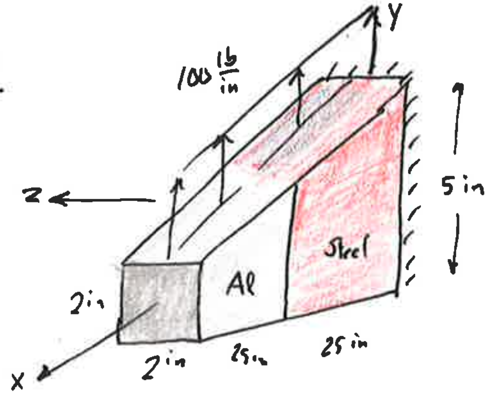
You can verify this surprising result with a piece of paper



- Load the tip ↓
- Observe the deflection of the centroid

- Where is the location where you can load the beam with no rotation.... stay tuned....

Ex 4.16



Find the vertical deflection

① Geometry Need h as a function of x. $h(0) = 5$ $h(50) = 2$

h is a linear function, thus $\frac{dh}{dx} = \text{constant}$

Integrate $\int_{h(0)}^{h(x)} dh = \int_0^x C\,dx$

$$h(x) - h(0) = Cx \quad \Rightarrow \quad \cancel{h(L)}^{2} = \cancel{h(0)}^{5} + C\cdot\cancel{L}^{50}$$

$$\frac{2-5}{50} = C = -\frac{3}{50}$$

$$\underline{h(x) = 5 - \frac{3}{50}x}$$

$$I_{zz} = \frac{1}{12}bh^3 = \frac{1}{12}\cdot 2\cdot\left(5 - \frac{3}{50}x\right)^3$$

② Loading

$$\frac{dV_y}{dx} = -P_y = -100\ \frac{\text{lbf}}{\text{in}} \quad \Rightarrow \quad \int_{V_y(L)}^{V_y(x)} dV_y = \int_L^x -100\,dx$$

$$V_y(x) - \cancel{V_y(L)}^{0} = -100x + 100\cancel{L}^{50}$$

$$\underline{V_y(x) = -100x + 5000}$$

$$\frac{dM_z}{dx} = -V_y = 100x - 5000$$

$$M_z(x) = \frac{100x^2}{2} - 5000x \Big|_L^x = \frac{100x^2}{2} - 5000x - \frac{100L^2}{2} + 5000L$$

③ Deflection

$I^*_{yz} = 0$, so only v is non-zero $(w(x) = 0)$

$$\frac{d^2v}{dx^2} = \frac{(M_z - M_z^T) I^*_{yy} + (M_y + M_y^T) I^*_{yz}}{E_1 (I^*_{yy} I^*_{zz} - I^{*2}_{yz})}$$

$$= \frac{M_z I^*_{yy}}{E_1 (I^*_{yy} I^*_{zz})} = \frac{M_z}{E_1 I^*_{zz}}$$

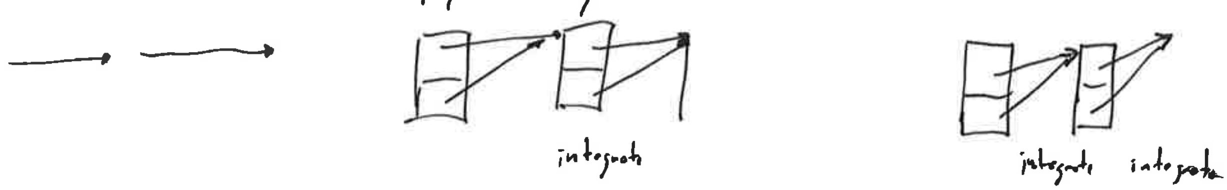
$$= \frac{50x^2 - 5000x + 125000}{E \, \frac{1}{12} \cdot 2 \cdot \left(5 - \frac{3}{50}x\right)^3}$$

Solve in two parts

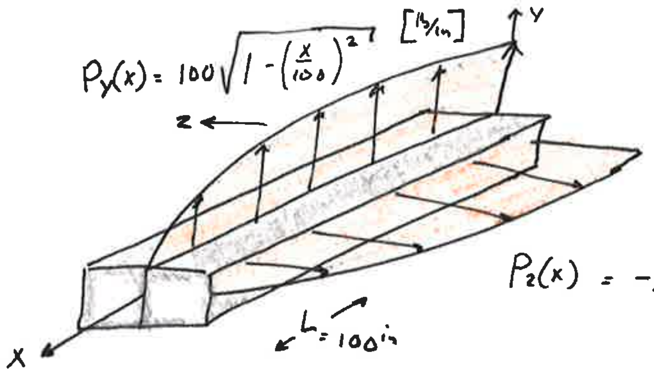
- $$v(x)_{steel} = \int_0^x \int_0^x \frac{d^2v}{dx^2} \, dx \, dx = \iint_0^x \frac{50x^2 - 5000x + 125000}{30 \times 10^6 \, \frac{1}{6} \left(5 - \frac{3}{50}x\right)^3}$$
- $$v(x)_{Al} = v(25)_{steel} + \iint_{25}^{x} \frac{50x^2 - 5000x + 125000}{10 \times 10^6 \, \frac{1}{6} \left(5 - \frac{3}{50}x\right)^3}$$

Can easily be solved in Excel

x h I_{zz} P_y V_y M_z E $\frac{d^2v}{dx^2}$ $\frac{dv}{dx}$ v



4.9



Elliptical lift distribution.

$$P_z(x) = -25\sqrt{1-\left(\frac{x}{100}\right)^2} \quad [\text{lb/in}]$$

① Geometry

$\bar{y}^* = 0 \quad \bar{z}^* = 0$ as drawn. from previous notes

$I^*_{yy} = 981 \text{ in}^4 \qquad I^*_{zz} = 526 \text{ in}^4 \qquad E_1 = 10 \times 10^6 \text{ psi}$

② Loading

$$P_y(x) = 100\sqrt{1-\left(\frac{x}{100}\right)^2} \qquad P_z(x) = -25\sqrt{1-\left(\frac{x}{100}\right)^2}$$

$$\frac{dV_y}{dx} = -P_y = -100\sqrt{1-\left(\frac{x}{100}\right)^2} \Rightarrow \int_{V_y(L)}^{V_y(x)} dV_y = \int_L^x -100\sqrt{1-\left(\frac{x}{100}\right)^2}\,dx$$

$$V_y = 5000 \operatorname{acos}\left(\frac{x}{100}\right) - \frac{x}{2}\sqrt{10000-x^2}$$

$$\frac{dV_z}{dx} = -P_z(x) = 25\sqrt{1-\left(\frac{x}{100}\right)^2}$$

$$V_z = -1250 \operatorname{acos}\left(\frac{x}{100}\right) + \frac{x}{8}\sqrt{10000-x^2}$$

$$\frac{dM_y}{dx} = -m_y(x) + V_z(x) = \uparrow =$$

$$M_y(x) = \int_L^x \left(-1250 \operatorname{acos}\left(\frac{x}{100}\right) + \frac{x}{8}\sqrt{10000-x^2}\right)dx$$

Not so easy!

$$\frac{dM_z}{dx} = -m_z + V_y = 5000 \operatorname{acos}\left(\frac{x}{100}\right) - \frac{x}{2}\sqrt{10000-x^2}$$

$$M_z = \int_L^x \rightarrow dx = \frac{30000\,x \operatorname{asin}\left(\frac{x}{100}\right) - \left(10000-x^2\right)^{3/2} + 30000\sqrt{10000-x^2} - 15000\pi x}{24}$$

Not so easy

What do we do?

(2a) Loading

Read the problem statement: "Determine the axial stresses and strains in the stringers and upper and lower skins at $\underline{x = 0}$"

We really only need $M_y(0)$ and $M_z(0)$!

$$M_y(0) = \int_L^0 V_z \, dx = \int_L^0 \left(-1250 \operatorname{acos}\left(\tfrac{x}{100}\right) + \tfrac{x}{8}\sqrt{10000 - x^2}\right) dx$$

$$= 83.3^{\text{ k lbf-in}}$$

$$M_z(0) = \int_L^0 V_y = \int_L^0 \left(5000 \operatorname{acos}\left(\tfrac{x}{100}\right) - \tfrac{x}{2}\sqrt{10000 - x^2}\right) dx$$

$$= -333^{\text{ k lbf-in}}$$

Alternatively, use the 0.4244 rule of elliptical wings.

$$M_{root_z} = 0.4244 \cdot \frac{L}{2} \cdot \frac{b}{2} = 0.4244 \tfrac{L}{2}\tfrac{b}{2} \quad \text{and} \quad L = \frac{b\pi}{4} L_0' = 2500\pi$$

$$= 0.4244 \frac{b\pi}{2 \cdot 8} L_0' \cdot 100 = 0.4244 \cdot \frac{100 \cdot \pi}{2 \cdot 8} \cdot 100 \cdot 100$$

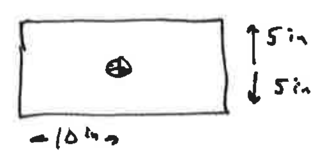
$$= \text{k lbf-in} \quad \checkmark$$

$$-333$$

$$M_{root_y} = 0.4244 \cdot \frac{b\pi}{8} L_0' \cdot \frac{b}{2} = 0.4244 \cdot \frac{100 \cdot \pi}{8} \cdot 25 \cdot 100$$

$$= 83.3^{\text{ k lbf in}}$$

(3) Stresses

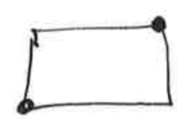


$$\sigma_{xx} = \frac{E}{E_1}\frac{M_z}{I_{zz}^*} y + \frac{E}{E_1}\frac{M_y z}{I_{yy}^*}$$

$$= 1 \; \frac{-333}{526} \cdot 5 \text{ in} + 1 \; \frac{83.3}{981} \, 10$$

Max when y and z are opposite signs

$$= \pm 2.3 \text{ ksi}$$



$$\sigma_{xx} = \frac{E}{E}\frac{M_z}{I_{zz}} y + \frac{E}{E_1}\frac{M_y z}{I_{yy}}$$

Steel

$$3 \cdot \frac{-333}{526} 5 + 3 \frac{83.3}{981} \cdot 10 = \pm 7 \text{ ksi}$$