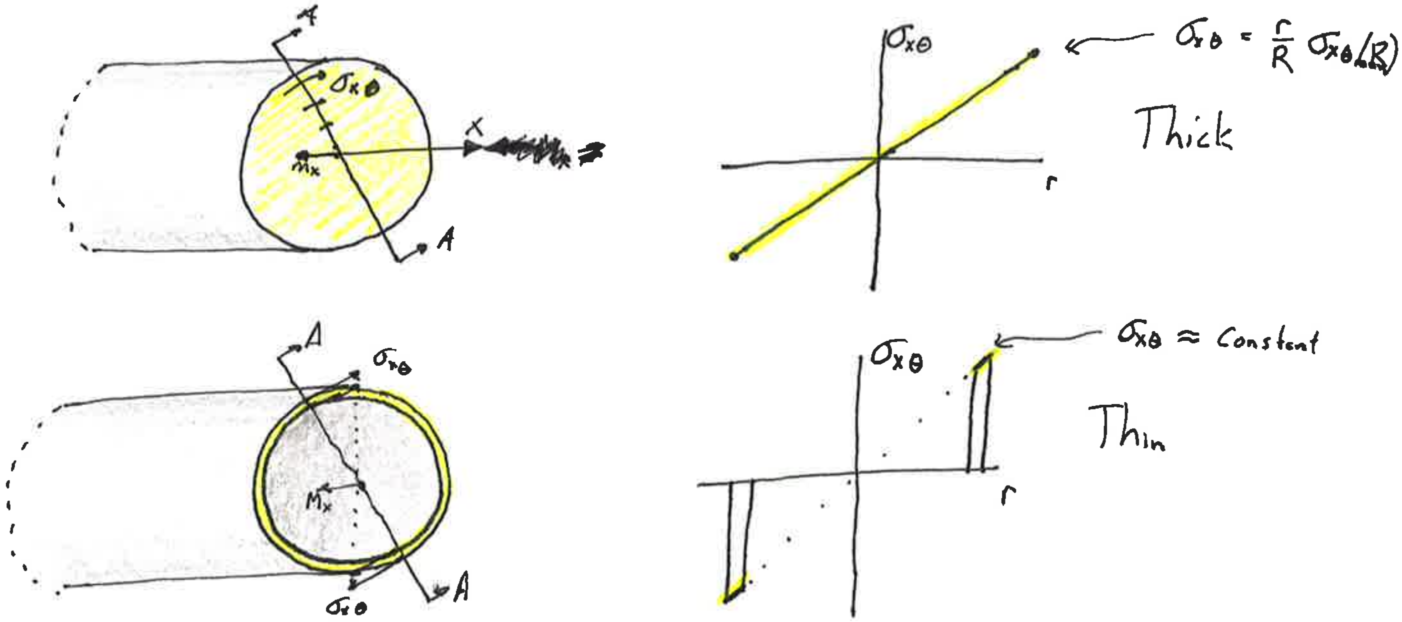
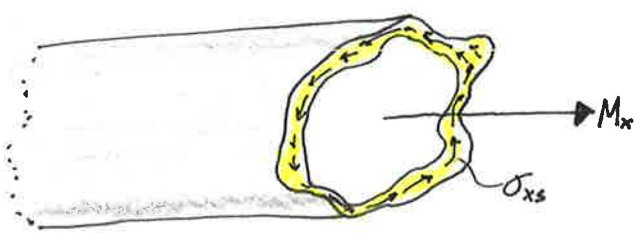
Thin Walled Closed Sections in Torsion

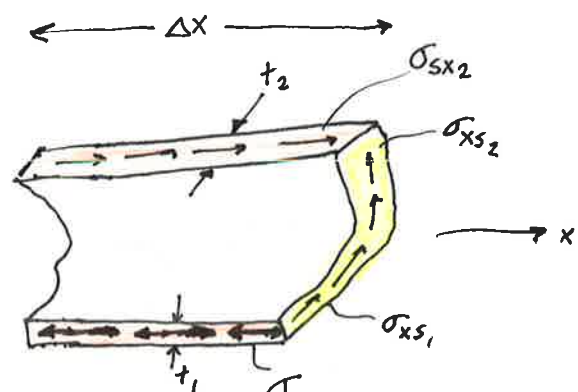
Thin wall ≡ stress is relatively constant across the wall thickness

Thick wall : stress in NOT constant across the wall thickness



Generic Thin Walled Closed Section





For no rotation $\sigma_{xs_1} = \sigma_{sx_1}$ and $\sigma_{xs_2} = \sigma_{sx_2}$ $\Rightarrow$ $\sum M = 0$ static

Sum forces in x-direction

$$\sum F_x = 0 = -\cancelto{+\sigma_{xs_1}}{\sigma_{sx_1}}(\Delta x)(t_1) + \cancelto{+\sigma_{xs_2}}{\sigma_{sx_2}}(\Delta x)(t_2)$$

Thus $\sigma_{xs_1} t_1 = \sigma_{xs_2} t_2 =$ constant for a given M_x !!

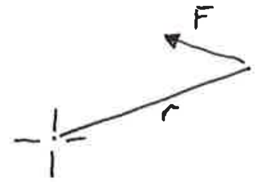
We call this constant

$$\boxed{\text{Shear flow} = q \equiv \sigma_{xs}\, t}$$

Units $[psi][in] = \left[\frac{lbf}{in}\right]$

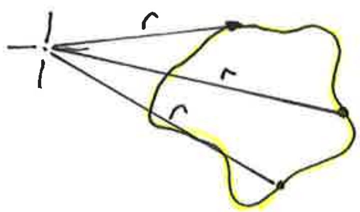
Torque about a point

$$M = r \times F$$

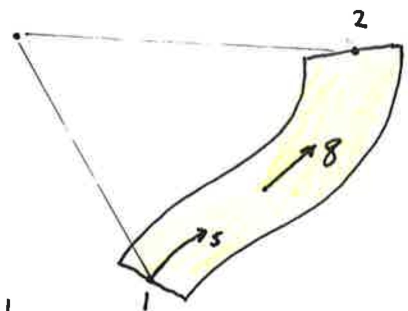


$$dM = r \times dF$$

Pick a section of our thin walled closed section with an arbitrary point for M



$\Rightarrow$



$F = q\,ds$

q is always in the s direction since thin walled

$$T_{12} = M_{12} = \int_1^2 r \times q\,ds$$

But $r = r(s)$

$$M = \int_1^2 r(s) \times q\,ds \qquad q \text{ is a constant}$$

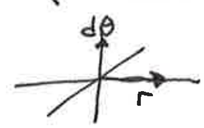
$$= q\int_1^2 r(s) \times ds$$

This looks complicated but remember that $ds = d\theta \times r$

$$M = q\int_1^2 r(s) \times (d\theta \times r)$$ and the vector triple product is

$$a \times (b \times c) = b(a\cdot c) - c(a\cdot b)$$

$$r \times (d\theta \times r) = d\theta\underbrace{(r\cdot r)}_{r^2} - r(r\cdot d\theta)$$



$r\cdot d\theta = 0$!

$$= d\theta\, r^2 - 0$$

and $r d\theta = ds$

$$M = q\int_1^2 r\,ds$$

The area of this differential piece is $\frac{1}{2} r\,ds = dA$

or $2dA = r\,ds$

$$= q\int_1^2 2dA$$

$$T_{12} = M_{12} = 2q\int_1^2 dA$$ all around section

See fig 4.20

$$T = 2q\bar{A} \Rightarrow \boxed{q = \frac{M_x}{2A}}$$

This is an important result. Shear flow depends on the applied moment and the cross section's area.

Ex:

A thin walled box section is subjected to 100 lbf-in. The interior area is 10 in². What is the shear flow?

$$q = \frac{M_x}{2A} = \frac{100\ \text{lbf-in}}{2 \cdot 10\ \text{in}^2} = \boxed{5\ \frac{\text{lbf}}{\text{in}} = q}$$

If the wall thickness is 0.010 inches at a point A, what is σ_{xs} at A?

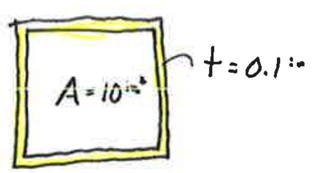
$$\sigma_{xs} = \frac{q}{t} = \frac{5\ \text{lbf}}{\text{in} \cdot 0.01\ \text{in}} = \boxed{500\ \text{psi} = \sigma_{xs}}$$

Have we specified the exact geometry yet?

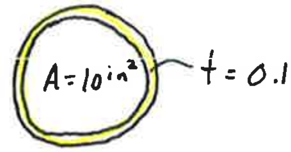
No

Ex: 4.20

~~Two~~ 3 cross sections are given. Determine the one with the lowest σ_{xs}.



①



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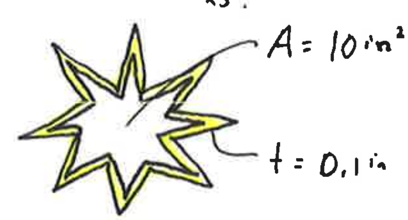


$$q = \frac{M_x}{2A} = \frac{M_x}{20\text{in}^2} \qquad q = \frac{M}{2A} = \frac{M_x}{20\text{in}^2} \qquad q = \frac{M_x}{20\text{in}} \quad \text{Same}$$

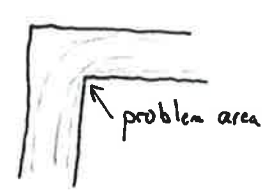
$$\sigma_{xs} = \frac{q}{t} = \frac{M_x}{2\ \text{in}^3} \qquad \sigma_{xs} = \frac{q}{t} = \frac{M_x}{2\ \text{in}^3} \qquad \sigma_{xs} = \frac{M_x}{2\text{in}^3} \quad \text{Same}$$

Theory says these are absolutely exactly the same.

Reality:

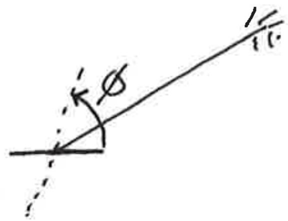
Stress concentrations

vs



think fluid flow

Torsional Deformation



Define a twist per unit length as

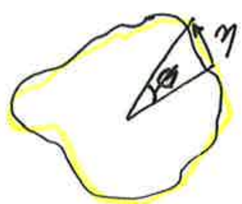
$$\Theta = \frac{d\phi}{dx} \implies d\phi = \Theta\, dx$$

$$\int d\phi = \int \Theta\, dx$$

$$\phi = \int \Theta\, dx$$

Remember back to the Kinematics section of the class

$$\varepsilon_{ij} = \frac{\partial u_j}{\partial i} + \frac{\partial u_i}{\partial j} \implies \varepsilon_{xs} = \frac{\partial u}{\partial s} + \frac{\partial \eta}{\partial x}$$



$\eta = r\phi$ = length of displacement due to rotation

And $\sigma_{xs} = G\,\varepsilon_{xs} \implies \varepsilon_{xs} = \frac{\sigma_{xs}}{G} = \frac{\partial u}{\partial s} + \frac{\partial \eta}{\partial x}$

$$= \frac{\partial u}{\partial s} + \frac{\partial}{\partial x}(r\phi)$$

$$= \frac{\partial u}{\partial s} + r\frac{\partial \phi}{\partial x} = \frac{\partial u}{\partial s} + r\Theta$$

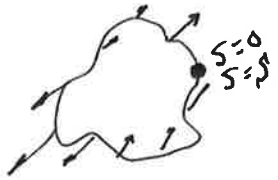
Also, $\sigma_{xs} = \frac{q}{t}$ so, $\frac{\sigma_{xs}}{G} = \frac{q}{Gt} = \overbrace{\frac{du}{ds} + r\Theta}$

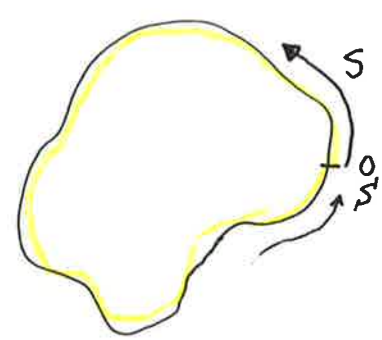
Solve for $\frac{du}{ds}$

$\frac{du}{ds} = \frac{q}{Gt} - r\Theta$ multiply by ds $du = \frac{q}{Gt}ds - r\Theta\, ds$

Integrate around S (i.e. the outside edge)

$$\oint_S du = \oint_S \frac{q}{Gt}\, ds - \oint_S r\Theta\, ds$$





$$\oint_S du = U(s=S) - U(s=0)$$

Same point!

$$\oint du = \underline{\underline{0}}$$

Thus

$$\underbrace{\oint_{S'} r\theta \, ds}_{\text{Remember this?}} = \oint_{S'} \frac{q}{Gt} ds$$

Remember this? $\theta \neq \theta(s)$ and is a constant in here

$$\theta \oint_{S'} r \, ds = \theta \cdot 2A$$

Thus,

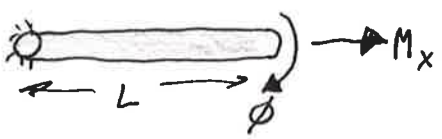
$$\boxed{\theta = \frac{1}{2A} \oint \frac{q}{Gt} ds}$$

Torsional Rigidity

spring constant

$$F = +Kx \Rightarrow K = \frac{+F}{x}$$

Similarly

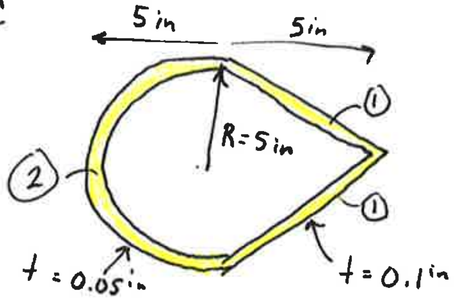


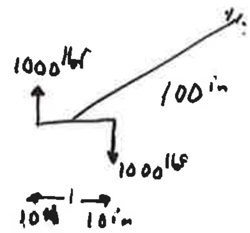
$$\theta \equiv \frac{d(\text{rotation})}{d(\text{Length})} = \frac{\phi}{L}$$

Torsional Rigidity $\equiv \boxed{TR = \frac{M_x}{\theta}}$

Think of a rotational spring constant

Ex: 4.19





part	G
1	5×10^6 psi
2	12×10^6 psi

① Loading

$$M_x = r \times F = -10^{in} \cdot 1000^{lbf} + 10^{in} \cdot (-1000^{lbf})$$

$$M_x = -20000^{lbf\,in}$$

② Geometry

$$\text{Area} = \text{◗} + \text{▷} = \frac{\pi R^2}{2} + \frac{h \cdot w}{2} = \frac{\pi\, 5^{in^2}}{2} + \frac{10^{in} \cdot 5^{in}}{2}$$

$$= 64.3^{in^2}$$

③ Shear flow

$$q = \frac{M_x}{2A} = \frac{-20000^{lbf\,in}}{2 \cdot 64.3^{in^2}} = -155.6 \frac{lbf}{in}$$

④ Shear Stress

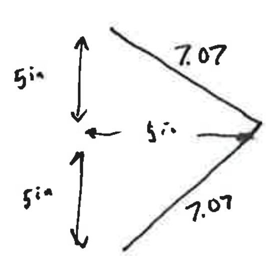
$$\sigma_{xs\,part\,1} = \frac{q}{t} = \frac{-155.6\,lbf}{in} \bigg| \frac{}{0.1\,in} = \boxed{-1.56 \text{ ksi}}$$

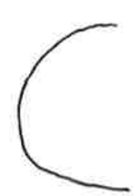
$$\sigma_{xs\,part\,2} = \frac{q}{t} = \frac{-155.6\,lbf}{in} \bigg| \frac{}{0.05\,in} = \boxed{-3.12 \text{ ksi}}$$

⑤ Twist

$$\theta = \frac{1}{2A}\oint \frac{q}{Gt}\,ds = \frac{1}{2A}\int_1 \frac{q_1}{G_1 t_1}\,ds + \frac{1}{2A}\int_2 \frac{q_2}{G_2 t_2}\,ds \qquad \text{but } q_1 = q_2 = q$$

$$= \frac{q}{2A} \oint \frac{1}{5\times10^6\,psi} \bigg| \frac{}{0.1\,in} \bigg| 14.14^{in} + \frac{q}{2A} \cdot \frac{1}{12\times10^6\,psi} \bigg| \frac{}{0.05}\ 15.7$$





$$C = \frac{\pi D}{2} = 15.7$$

$$\theta = \frac{-155.6\ \text{lbf}}{\text{in}} \cdot \frac{1}{2} \cdot \frac{1}{64.3\ \text{in}^2}\left(\frac{14.14\ \text{in}}{5\times10^6\ \text{psi} \cdot 0.1\ \text{in}} + \frac{15.7\ \text{in}}{12\times10^6\ \text{psi} \cdot 0.05\ \text{in}}\right)$$

$$= -0.000066\ \frac{\text{rad}}{\text{in}}$$

$$\text{Total twist} = \phi = \int \theta\, dx = \theta \cdot L = \boxed{-0.0066\ \text{rad} = \phi}$$

6) Torsional rigidity

$$TR = \frac{M_x}{\theta} = \frac{-20000\ \text{lbf in}}{-0.000066\ \text{rad/in}} = \boxed{3.0\times10^8\ \frac{\text{lbf in}^2}{\text{rad}}}$$

Compare (Roughly) to broken section

$$J_1 \approx \frac{st^3}{3} = \frac{14.14\ \text{in} \cdot 0.1^3\ \text{in}^3}{3} = 0.0047\ \text{in}^4 \qquad J_2 = \frac{15.7\ \text{in} \cdot 0.05^3\ \text{in}^3}{3} = 0.00065\ \text{in}^4$$

$$TR = GJ \approx \Sigma GJ = (5\times10^6\ \text{psi})(0.0047\ \text{in}^4) + (12\times10^6\ \text{psi})(0.00065\ \text{in}^4)$$

$$= 31416.7\ \frac{\text{lbf in}^2}{\text{rad}}$$

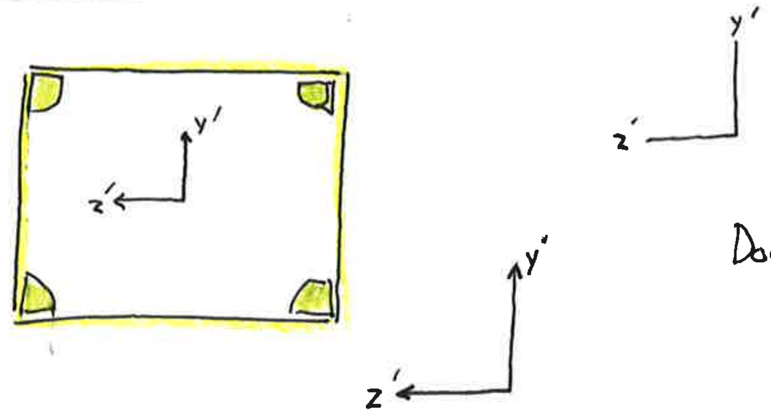
$$\theta = \frac{M_x}{TR} = \frac{-20000\ \cancel{\text{lbf in}}\ \text{rad}}{31416.7\ \cancel{\text{lbf}}\ \text{in in}} = -0.636\ \frac{\text{rad}}{\text{in}}$$

$$\phi \equiv \text{total twist} = L\cdot\theta = -63\ \text{rad}\ !!$$

$$= 10\ \text{twists}$$

TR is .01% of closed section.

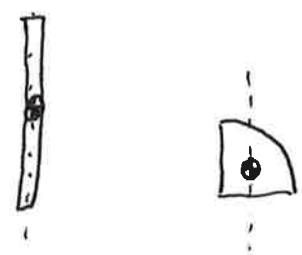
Computing Modulus Weighted I_{yz} I_{yy} I_{zz}



Does the reference frame matter?

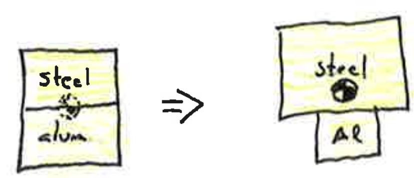
No

① Calculate $I_{y_0y_0}$ of each part about its centroid



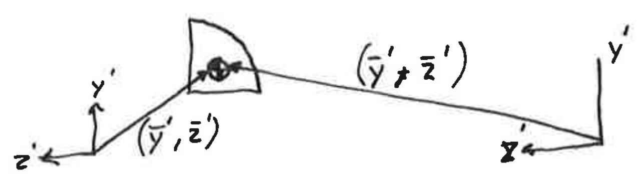
② Modulus weighted Area

$$A^* = \sum \frac{E}{E_1} A$$



As if the geometry were scaled by E

③ distance to centroid from reference frame y' z'



④ Total MoI about cross sections' modulus weighted centroid

$$\underbrace{\underbrace{I_{yy}^* = \frac{E}{E_1}\underbrace{\left(\overbrace{I_{y_0y_0}}^{\text{mod about self}} + \bar{z}_i'^{\,2} A_i\right)}_{\text{modulus about } y'z' \text{ frame}}}_{\text{weighted modusu}} - \underbrace{\bar{z}'^{*\,2} A^*}_{\text{shift to modulus weighted centroid}}}_{\text{MW Centroid } I_{yy}^*}$$

$\bar{z}_i'$ is the z distance from the y'z' frame to the part's centroid

$\bar{z}'^*$ is the z distance from the y'z' frame to the MW centroid