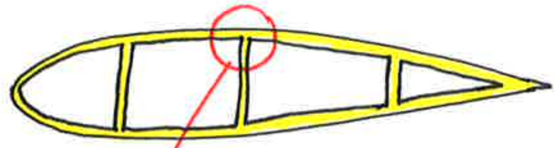
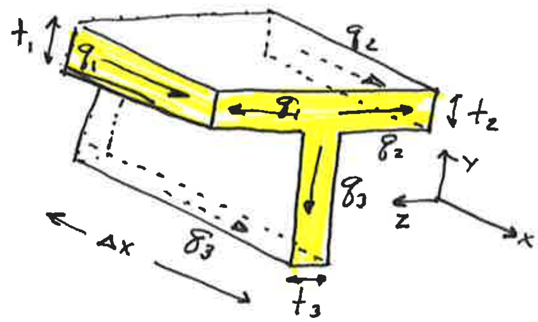
Multicell closed section Torsion (Bredt method)

Many aerospace structures use more than one cell



Take a section of an interface between two cells



Force is $\sigma_{xs} \cdot t \cdot \Delta x = \frac{q}{t} \cdot t \cdot \Delta x = q \Delta x$

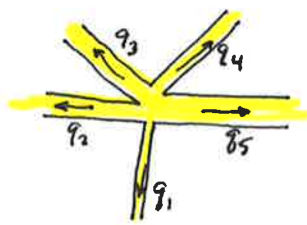
$$\Sigma F_x = 0 = q_1 \Delta x + q_2 \Delta x + q_3 \Delta x$$

$\Delta x \neq 0$ so $\boxed{q_1 + q_2 + q_3 = 0}$

Think of fluid flow, where the mass flow rate summed to zero at a junction.

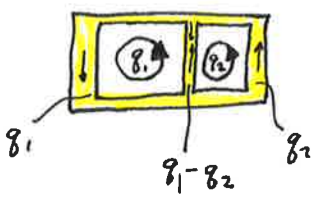
Conservation of shear flow

In general,



$$\boxed{\Sigma q_i = 0}$$

We also know that the torque is the summation of all shear flows and areas



$$M_x = \Sigma T = \Sigma 2 q_i A_i \Rightarrow \boxed{M_x = 2 \Sigma q_i A_i}$$

Recall that $M_x = 2 q \bar{A}$ for 1 cell

For an n-cell structure, we have $q_1 \cdots q_n$ (n shear flows). Need n equations!

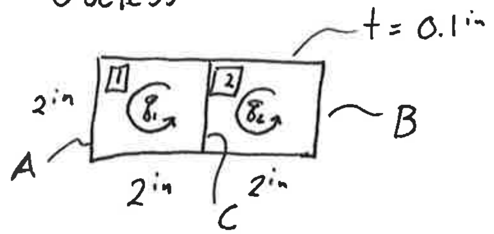
1) $M_x = 2 \Sigma q_i A_i$ (this is one equation)

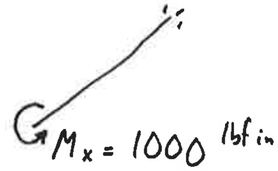
We also know that the cells twist together. $\text{Twist}_1 = \text{Twist}_2 = \cdots = \text{Twist}_n$

$$\theta = \frac{1}{2A_1} \oint \frac{q}{G_1 t_1} ds_1 = \frac{1}{2A_2} \oint \frac{q}{G_2 t_2} ds_2 = \cdots = \frac{1}{2A_n} \oint \frac{q}{G_n t_n} ds_n$$

Each set gives 1 equation $\Rightarrow$ n-1 more equations! Solved!

Example "Useless"





$G = 10\times10^6$ psi

① Loading: $M_x = 1000$ lbf in $\circlearrowleft q_1$ $\circlearrowleft q_2$

② Section properties:

$$A_1 = 4\ in^2 \qquad A_2 = 4\ in^2$$

③ Applied Torque

$$M_x = 2\sum q_i A_i = 2(q_1 4 + q_2 4) = \underline{8q_1 + 8q_2 = 1000\ \text{lbf in}}$$

④ Twist

$$\theta_1 = \frac{1}{2A_1}\oint \frac{q}{G_1 t_1} ds_1 = \underbrace{\frac{1}{2A}\frac{1}{G_1 t_1}}_{\frac{1}{2\cdot4}\frac{1}{10\times10^6\cdot0.1}}\Big(\underbrace{3\cdot2\cdot q_1}_{\boxed{\circlearrowleft q_1}} + \underbrace{2(q_1 - q_2)}_{q_1\circlearrowright | \circlearrowleft q_2}\Big) = \frac{1}{2AG_1t_1}(4\cdot2\cdot q_1 - 2q_2)$$

$$= 1\times10^{-6} q_1 - 2.5\times10^{-7} q_2$$

$$\theta_2 = \frac{1}{2A_2}\oint \frac{q}{G_2 t_2} ds_2 = \frac{1}{2AG_2t_2}\Big(\underbrace{2(q_2 - q_1)}_{|} + \underbrace{3\cdot2\cdot(q_2)}_{\sqsupset}\Big) = \frac{1}{2AG_2t_2}(\underbrace{4\cdot2\cdot q_2}_{\square} - 2q_1)\ \ -|$$

$$= 1\times10^{-6} q_2 - 2.5\times10^{-7} q_1$$

$$\theta_1 = \theta_2 = 10q_1 - 2.5q_2 = 10q_2 - 2.5q_1 \Rightarrow \quad 7.5q_1 - 7.5q_2 = 0$$

⑤ Matrix

$$\begin{bmatrix} 8 & 8 \\ 7.5 & -7.5 \end{bmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1000 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{\begin{bmatrix} -7.5 & -8 \\ -7.5 & 8 \end{bmatrix}}{-120}\begin{pmatrix} 1000 \\ 0 \end{pmatrix} = \boxed{\begin{matrix} +62.5 = q_1 \\ +62.5 = q_2 \end{matrix}}$$

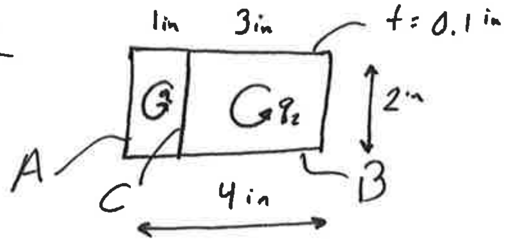
⑥ $\sigma_{A_{xs}} = \frac{q_1}{t} = \frac{62.5}{0.1} = 625$ psi $\sigma_{B_{xs}} = \frac{62.5}{0.1} = 625$ psi

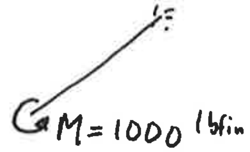
$\phi \approx 0.26°$

$$\sigma_{C_{xs}} = \frac{q_1 - q_2}{t} = 0$$

The perfectly centered center web is useless for torsion!

Example





Offset web

① Loading $M_x = 1000$ lbf·in

② Section Properties

$$A_1 = 2 \text{ in}^2 \qquad A_2 = 6 \text{ in}^2$$

③ Applied Torque

$$M_x = 2 \sum q_i A_i = 4 q_1 + 12 q_2$$

④ Twist

$$\theta_1 = \frac{1}{2A_1} \oint \frac{q}{G_1 t_1} ds_1 = \frac{1}{2A_1 G t}\left(6 q_1 - 2 q_2\right)$$

(□ − 1)

$$\theta_2 = \frac{1}{2A_2} \oint \frac{q}{G_2 t_2} ds_2 = \frac{1}{2A_2 G t}\left(10 q_2 - 2 q_1\right)$$

(▭ − 1)

$\Rightarrow \theta_1 = \theta_2$

$$\frac{6 q_1 - 2 q_2}{2} = \frac{10 q_2 - 2 q_1}{6}$$

$$3 q_1 - q_2 = \frac{10}{6} q_2 - \frac{1}{3} q_1$$

⑤ Matrix

$$\begin{bmatrix} 4 & 12 \\ 3.333 & -2.667 \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1000 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{\begin{pmatrix} -2.667 & -12 \\ -3.333 & 4 \end{pmatrix} \begin{pmatrix} 1000 \\ 0 \end{pmatrix}}{-50.667} \Rightarrow \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 52.63 \\ 65.78 \end{pmatrix}$$

⑥ Stresses

$$\sigma_{A_{xs}} = \frac{q}{t} = 526.3 \text{ psi} \qquad \sigma_{B_{xs}} = 657.8 \text{ psi} \qquad \sigma_{C_{xs}} = \frac{q_1 - q_2}{t} = -131 \text{ psi}$$

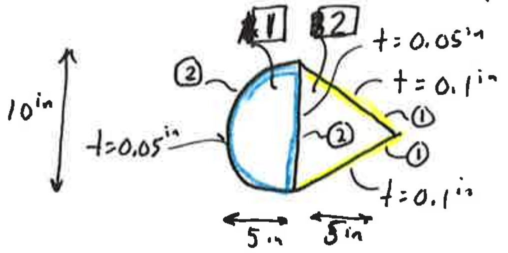
⑦ Twist

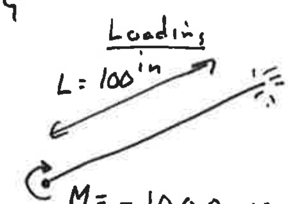
$$\theta = \frac{1}{2A_1 G t}\left(6 q_1 - 2 q_2\right) = 4.6 \times 10^{-5} \ \frac{\text{rad}}{\text{in}}$$

$$\boxed{\phi = \theta L = 0.26^\circ}$$

To minimize the difference in sheer stresses, we want ~~equally Area~~ equal Area cells of an odd number

4.22 Add a vertical web to 4.19





$M = -1000 \cdot 10 \cdot 2 = -20000$ lbf in

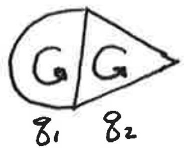
$G_1 = 5 \times 10^6$ psi
$G_2 = 12 \times 10^6$ psi

① Geometry

$A_1 = \frac{1}{2}\pi r^2 = 39.27 \text{ in}^2$ $\qquad A_2 = \frac{1}{2} 5^{in} \cdot 10^{in} = 25 \text{ in}^2$

$\sqrt{5^2+5^2} = 7.07^{in}$

② Shear flow



③ Equations

2 unknowns: $q_1 \; q_2$

2 equations: $M_x = 2q_1A_1 + 2q_2A_2$ and $\theta_1 = \theta_2$

$= 78.54 q_1 + 50 q_2$

④ Twist

$$\theta_1 = \frac{1}{2A_1}\oint \frac{q}{G_1 t_1} ds = \frac{q_1 \frac{1}{2}\pi \cdot 10 + q_1 10 - q_2 10}{2 \cdot 39.27 \cdot 12\times10^6 \cdot 0.05} = 5.455\times10^{-7} q_1 - 2.1221\times10^{-7} q_2$$

$$\theta_2 = \frac{1}{2A_2}\oint \frac{q}{G_2 t_q} ds = \frac{(q_2 - q_1)10}{2 \cdot 25 \cdot 12\times10^6 \cdot 0.05} + \frac{q_2 \cdot 2 \cdot 7.07}{2 \cdot 25 \cdot 5\times10^6 \cdot 0.1} = -3.33\times10^{-7} q_1 + 8.99\times10^{-7} q_2$$

$$\theta_1 = \theta_2 \Rightarrow \quad 8.788\times10^{-7} q_1 - 1.111\times10^{-6} q_2 = 0$$

⑤ Matrix

$$\begin{bmatrix} 78.54 & 50 \\ 8.788 & -11.11 \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} -20000 \\ 0 \end{pmatrix}$$

Gauss elimination

78.54	50	-20000
0	-16.71	+2237.8

$q_2 = -133.96$
$q_1 = -169.4$

⑥ Stress

$\sigma_{C_{xs}} = \frac{q_1}{t} = \frac{-169.4}{0.05} = -3.39$ ksi $\qquad \sigma_{>_{xs}} = \frac{q_2}{t} = -1.34$ ksi

$\sigma_{I_{xs}} = \frac{q_1 - q_2}{t} = -0.71$ ksi

⑦ Twist

$\theta = 5.455\times10^{-7} q_1 - 2.122\times10^{-7} q_2 = -0.000064 \frac{rad}{in}$

$\phi =$	$-0.366°$

⑧ $TR = \frac{M_x}{\theta} = 3.12\times10^8 \frac{lbf\,in^2}{rad}$

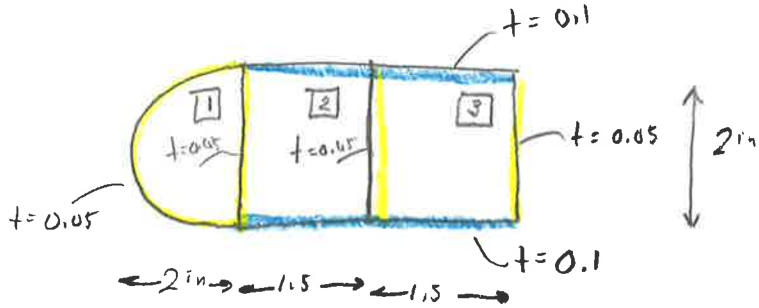
Compare with 4.19

$TR = 3.0\times10^8$

Not much impact!

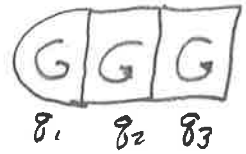
4.24 Add web to Example 4.6

$G = 4\times10^6$ psi



① Loading

$$M_x = 1000 \text{ lbf in}$$



② Section properties

$$A_1 = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi = 1.57$$

$$A_2 = 1.5\cdot 2 = 3 \qquad A_3 = A_2 = 3$$

③ Applied Torque

$$M_x = 2q_iA_i = 2\cdot1.57\cdot q_1 + 2\cdot3\cdot q_2 + 2\cdot3\cdot q_3$$

$$= 3.1415\,q_1 + 6\,q_2 + 6\,q_3$$

④ Twist

$$\theta_1 = \frac{1}{2A}\oint\frac{q}{Gt}ds = \frac{1}{3.1415}\left(q_1\,\frac{\frac{1}{2}\pi\cdot2 + 2}{4\times10^6\cdot0.05} - q_2\left(\frac{2}{4.0\times10^6\cdot0.05}\right)\right)$$

$$= 8.18\times10^{-6}\,q_1 - 3.18\times10^{-6}\,q_2$$

$$\theta_2 = \frac{1}{2A}\oint\frac{q}{Gt}ds = \frac{1}{6}\Big(-q_1\,\frac{2}{4\times10^6\cdot0.05} + q_2\,\frac{2+2}{4\times10^6\,0.05} + q_2\,\frac{3}{4\times10^6\cdot0.1} - q_3\,\frac{2}{4\times10^6\,0.05}$$

$$-1.667\times10^{-6}\,q_1 + 4.58\times10^{-6}\,q_2 - 1.667\times10^{-6}\,q_3$$

$$\theta_3 = \frac{1}{2A}\oint\frac{q}{Gt}ds = \frac{1}{6}\left(-q_2\,\frac{2}{4\times10^6\,0.05} + q_3\,\frac{3}{4\times10^6\cdot0.1} + q_3\,\frac{4}{4\times10^6\,0.05}\right)$$

$$-1.667\times10^{-6}\,q_2 + 4.583\times10^{-6}\,q_3$$

$\theta_1 = \theta_2 \Rightarrow$

θ_1	8.18	+ 1.667	= 9.847
$-\theta_2$	−3.18	− 4.58	= −7.76
	0	+ 1.667	1.667

$\Rightarrow 0 = 9.847\, q_1 - 7.76\, q_2 + 1.667\, q_3$

$\theta_2 = \theta_3 \Rightarrow \theta_2 - \theta_3 : -1.667\, q_1 + \underbrace{(4.58 + 1.667)}_{6.247} q_2 + \underbrace{(-1.667 - 4.583)}_{-6.247} q_3$

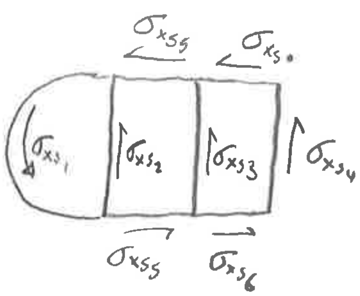
$-1.667\, q_1 + 6.247\, q_2 - 6.247\, q_3 = 0$

(5) Matrix

$$\begin{bmatrix} 3.1415 & 6 & 6 \\ 9.847 & -7.76 & 1.667 \\ -1.667 & 6.247 & -6.247 \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 1000 \\ 0 \\ 0 \end{pmatrix}$$

$q_1 = 49.9 \qquad q_2 = 76.9 \qquad q_3 = 63.6$

(6) Stresses



$\sigma_{xs_1} = \frac{q}{t} = \frac{49.9}{0.05} = 1000 \text{ psi}$

$\sigma_{xs_2} = \frac{q_1 - q_2}{t} = -540 \text{ psi}$

$\sigma_{xs_5} = \frac{q_2}{t} = \frac{76.9}{0.1} = 769 \text{ psi}$

$\sigma_{xs_3} = \frac{q_2 - q_3}{t} = 266 \text{ psi}$

$\sigma_{xs_6} = \frac{q_3}{t} = 636 \text{ psi}$

$\sigma_{xs_4} = \frac{q_3}{t} = 1272 \text{ psi}$

(7) Twist

$\theta = 8.18\times10^{-6}\, q_1 - 3.18\times10^{-6}\, q_2 = 0.000164 \frac{\text{rad}}{\text{in}}$

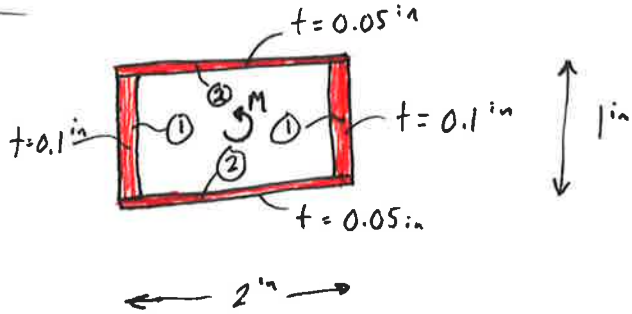
(8) TR

$TR = \frac{M_x}{\theta} = \frac{1000 \text{ lbf in}}{0.000164 \text{ rad/in}} = 6.11\times10^6 \frac{\text{lbf in}^2}{\text{rad}}$

Compared with  in problem 4.6

The TR is 2% higher.

Ex. 4.5



$G_1 = 3.75 \times 10^6$ psi

$G_2 = 12.0 \times 10^6$ psi

① Loading

$$M_x = 500^{lbf} \cdot 2^{in} = 1000^{lbf\text{-}in}$$

② Section properties

$$\text{Area} \approx 2 \cdot 1 = 2^{in^2}$$

③ Shear flow

$$q = \frac{M_x}{2A} = \frac{1000 \text{ lbf in}}{2 \cdot 2 \text{ in}^2} = 250 \frac{\text{lbf}}{\text{in}}$$

④ Stress

$$\sigma_{xs_1} = \frac{q}{t} = \frac{250 \frac{\text{lbf}}{\text{in}}}{0.1 \text{ in}} = \boxed{2500 \text{ psi} = \sigma_{xs_①}}$$

$$\sigma_{xs_2} = \frac{q}{t} = \frac{250 \frac{\text{lbf}}{\text{in}}}{0.05 \text{ in}} = \boxed{5000 \text{ psi} = \sigma_{xs_②}}$$

⑤ Twist

$$\theta = \frac{1}{2A}\oint \frac{q}{Gt} ds = \frac{1}{2A} \sum \frac{q_i}{G_i t_i} s_i = \frac{q}{2A} \sum \frac{s_i}{G_i t_i}$$

Notice:

$$\theta = \frac{1}{2A} q \oint \frac{1}{Gt} ds$$

$$= \frac{1}{2A} \frac{M_x}{2A} \oint \frac{ds}{Gt}$$

$$= \frac{M_x}{4A^2} \oint \frac{ds}{Gt}$$

Area squared!!

$$= \frac{250 \frac{\text{lbf}}{\text{in}}}{2 \cdot 2 \text{ in}^2} \left(\frac{2 \text{ in}}{12 \times 10^6 \text{ psi} \cdot 0.05 \text{ in}} \cdot 2 + \frac{1 \text{ in}}{3.75 \times 10^6 \text{ psi} \cdot 0.1 \text{ in}} \cdot 2 \right)$$

$$= 7.5 \times 10^{-4} \frac{\text{rad}}{\text{in}}$$

$$\phi = \theta L = 7.5 \times 10^{-4} \frac{\text{rad}}{\text{in}} \cdot 100 \text{ in} \cdot \frac{180°}{\pi \text{ rad}} \; (57.3) = \boxed{4.3° = \phi}$$

⑥ TR

$$TR = \frac{M_x}{\theta} = \frac{1000 \text{ lb in}}{7.5 \times 10^{-4} \text{ rad/in}} = \boxed{1.33 \times 10^6 \frac{\text{lbf in}^2}{\text{rad}}}$$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