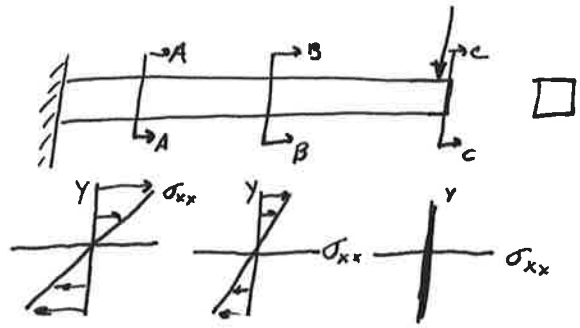
Shear in advanced beams

We previously learned how to find the axial stress in an advanced beam

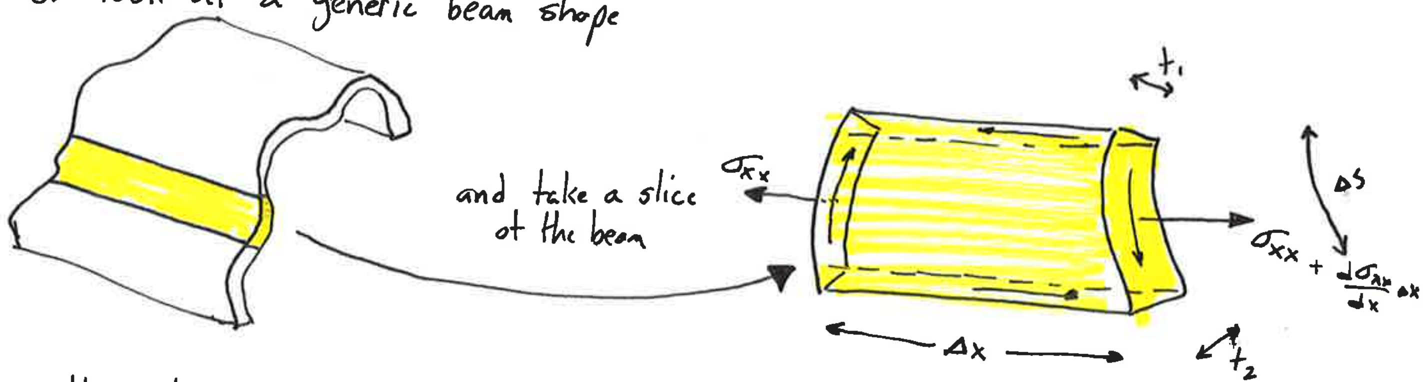
$$\sigma_{xx} = \frac{P}{A} - \frac{M_z I_{yy} + M_y I_{yz}}{I_{yy} I_{zz} - I_{yz}^2} y + \frac{M_y I_{zz} + M_z I_{yz}}{I_{yy} I_{zz} - I_{yz}^2} z$$

Since M_y, M_z, P and possibly even I_{yy}, I_{yz}, I_{zz} vary with location x, y, z

σ_{xx} is variable



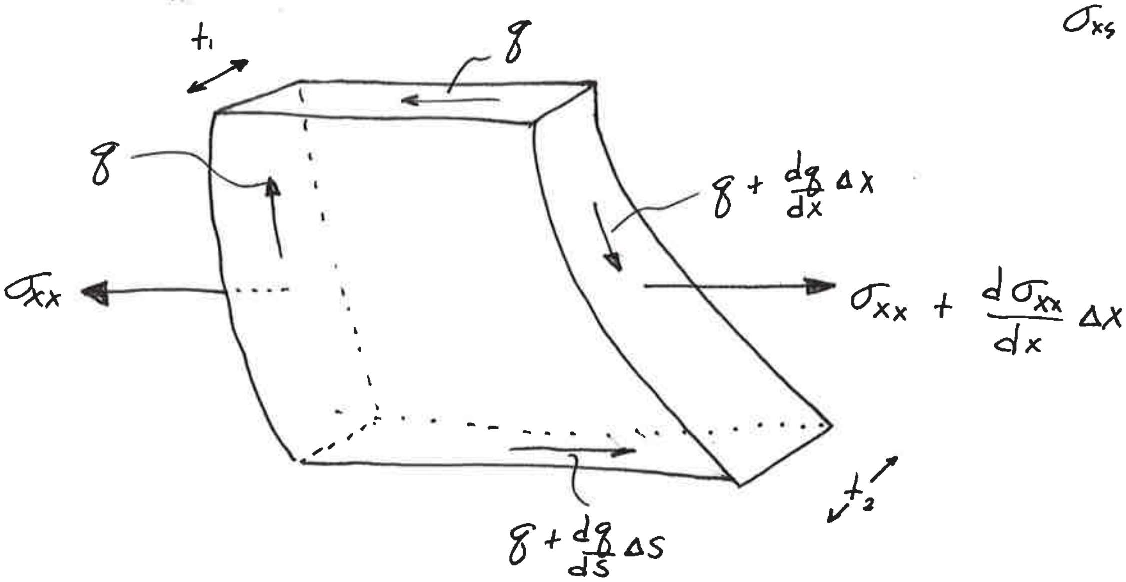
Let us look at a generic beam shape



Rather than track the shear stresses directly, use the <u>shear flow concept</u>

$$\sigma_{xs} = \frac{q}{t}$$

q



Summation of forces in x-direction

$dF = \sigma_{xs}\, dA$

$dA = t\, ds$

$$\sum F_x = 0 = \int^{\Delta s}\left(\sigma_{xx} + \frac{d\sigma_{xx}}{dx}\Delta x\right)\left(t_0 + \frac{dt}{dx}\Delta x\right) ds$$

$$-\int^{\Delta s}(\sigma_{xx})(t_0)\, ds$$

$$+\int^{\Delta x}\left(q + \frac{dq}{ds}\Delta s\right) dx$$

$$-\int^{\Delta x} q\, dx$$

Need to account for thickness change with respect to x

Cancel like terms ($\int \sigma_{xx} t_0\, ds - \int \sigma_{xx} t_0\, ds = 0$)

$$0 = \int^{\Delta s}\left(\frac{d\sigma_{xx}}{dx}\Delta x\right) t_0\, ds + \underbrace{\int \frac{d\sigma_{xx}}{dx}(\Delta x)^2 \frac{dt}{dx}\, ds}_{\text{when } \frac{dt}{dx}=0\text{, this term goes away}} + \int^{\Delta x} \frac{dq}{ds}\Delta s\, dx$$

As Δx and $\Delta s \to 0$ (or $\Delta x = dx$)

$$\frac{d\sigma_{xx}}{dx} t + \frac{dq}{ds} = 0 \implies \boxed{\frac{dq}{ds} = -\frac{d\sigma_{xx}}{dx} t}$$

The shear flow change in s depends on the ~~shear~~ axial stress change in x (and thickness)

We already derived σ_{xx}, so plug in and integrate

$$\int_{q_0}^{q(s)} \frac{dq}{\cancel{ds}}\cancel{ds} = \int_{s_0}^{s} -\frac{d\sigma_{xx}}{dx} t\, ds$$

$$q(s) = q_0 - \int_{s_0}^{s} \frac{d}{dx}(\sigma_{xx})\, t\, ds$$

$$q(s) = q_0 - \int_{s_0}^{s} \frac{d}{dx}\left(\frac{E}{E_1}\frac{P^T}{A^*} - \frac{E}{E_1}\left(\frac{(M_z - M_z^T) I_{yy}^* + (M_y - M_y^T) I_{yz}^*}{I_{yy}^* I_{zz}^* - I_{yz}^{*2}}\right) y \right.$$

$$+ \frac{E}{E_1}\left(\frac{(M_y - M_y^T) I_{zz}^* + (M_z - M_z^T) I_{yz}^*}{I_{yy}^* I_{zz}^* - I_{yz}^{*2}}\right) z$$

$$\left. - E\alpha\Delta T \right) t \, ds$$

In general, M_z and I_{zz}^* could vary with x making this a tedious chain rule $\frac{d}{dx}$ calc.
If the beam properties are constant (but M_z loading terms are not constant)

$$q(s) = q_0 - \int_{s_0}^{s} \left(\frac{E}{E_1 A^*}\frac{dP^T}{dx} - \frac{E}{E_1}\frac{y}{I_{yy}^* I_{zz}^* - I_{yz}^{*2}}\left(I_{yy}^* \frac{d}{dx}(M_z - M_z^T) + I_{yz}^* \left(\frac{d}{dx}(M_y - M_y^T)\right)\right) \right.$$

$$- \frac{E}{E_1}\frac{z}{I_{yy}^* I_{zz}^* - I_{yz}^{*2}}\left(I_{zz}^* \frac{d}{dx}(M_y - M_y^T) + I_{yz}^* \frac{d}{dx}(M_z - M_z^T) \right)$$

$$\left. - E\alpha \frac{dT}{dx} \right) t \, ds$$

But remember that the differential equations of beams are

$$\frac{dM_z}{dx} = -m_y \quad \text{and others (et.c)} \quad \text{and} \quad M_z^T = \int E\alpha\Delta T y \, dA$$

Such that

$$\frac{d}{dx}(M_z - M_z^T) = -m_y - \frac{d}{dx} M_z^T = -m_y - \underbrace{\int E\alpha \frac{dT}{dx} y \, dA}_{m_z^T}$$

Shear flow

$$q(s) = q(0) - p^T \frac{A_s^*}{A^*} + Q_z^* \left(\frac{(-m_z - V_y - m_z^T) I_{yy}^* + (-m_y + V_z + m_y^T) I_{yz}^*}{I_{yy}^* I_{zz}^* - I_{yz}^{*2}} \right)$$

$$- Q_y^* \left(\frac{(-m_y + V_z + m_y^T) I_{zz}^* + (-m_z - V_y - m_z^T) I_{yz}^*}{I_{yy}^* I_{zz}^* - I_{yz}^{*2}} \right)$$

$$+ E\alpha \int_0^s \frac{dT}{dx} t \, ds$$

$$p^T = \int_A E\alpha \frac{dT}{dx} dA$$

$$m_y^T = \int_A E\alpha \frac{dT}{dx} z \, dA$$

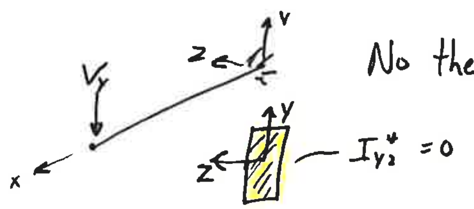
$$m_z^T = \int_A E\alpha \frac{dT}{dx} y \, dA$$

$$A_s^* = \int_0^s \frac{E}{E_1} t \, ds$$

$$Q_y^* = \int_0^s \frac{E}{E_1} z t \, ds$$

$$Q_z^* = \int_0^s \frac{E}{E_1} y t \, ds$$

Ex



No thermal

$I_{yz}^* = 0$

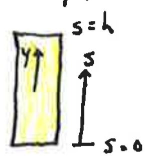
$$q(s) = q(0) + \frac{Q_z^*(-V_y)}{I_{zz}^*} = q(0) - \frac{V_y Q_z^*}{I_{zz}^*}$$

$$Q_z = \int_0^s y t \, ds$$

make $s=0$ the top of beam

From traction theory, q on top/bottom face $= 0$

$$Q_z = \int_0^s y(s) t(s) ds$$



$y = s - h/2$

t = constant = w

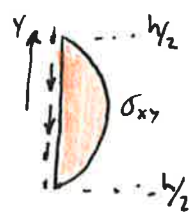
$$= \int_0^s (s - h/2) w \, ds = \frac{ws^2}{2} - \frac{wh}{2} s \Big|_0^s =$$

$$= \frac{ws^2}{2} - \frac{whs}{2}$$

$$q(s) = \cancel{q_0}^{0} - \frac{-V_y}{1/12\, wh^3}\left(\frac{ws^2}{2} - \frac{whs}{2}\right) = \frac{6V_y s^2}{h^3} - \frac{6V_y s}{h^2} = \frac{6V_y}{h}\left(\frac{s^2}{h^2} - \frac{s}{h}\right)$$

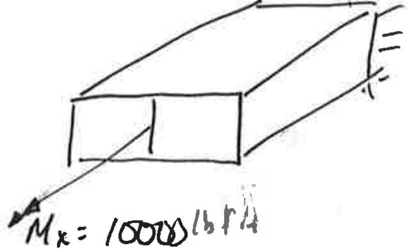
$$\sigma_{xy} = \frac{q}{t} = \frac{6V_y}{wh}\left(\frac{s^2}{h^2} - \frac{s}{h}\right)$$

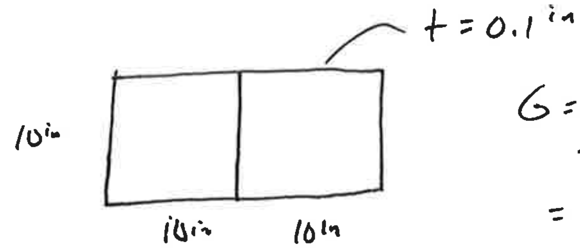
$\sigma_{xy_{max}} = 1.5 \frac{V_y}{wh}$ or 50% higher than constant.



For thick beams, σ_{xy} may not be large, but for aerospace vehicles with thin parts, σ_{xy} can be quite large

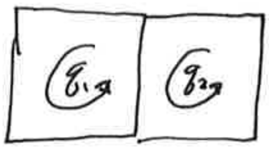
4.21



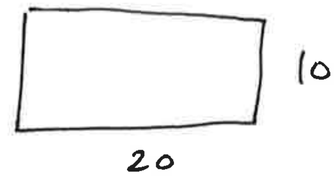


$$G = \frac{E}{2(1+\nu)} = \frac{10\times10^6}{2(1.3)} = 3.8\times10^6$$

Can you solve this easily?



$$q_1 = q_2$$
$$q_1 - q_2 = 0 \Rightarrow$$



$$M = 2qA = 400\,q = 10000$$

$$q = 25$$

$$\theta = \frac{1}{2A}\oint \frac{q}{Gt}\,ds = \frac{1}{400}\,\frac{25}{3.8\times10^6\cdot 0.1} = 1.64\times10^{-7}\ \frac{\text{rad}}{\text{in}}$$

$$\sigma_{xs} = \frac{q}{t} = \frac{25}{0.1} = 250\ \text{psi}$$

Hard Way

$$M = 2q_1\cdot 100 + 2q_2\cdot 100$$

$$\theta_1 = \frac{1}{200}\oint \frac{q}{Gt}\,ds = \frac{1}{200}\left(\frac{q_1\cdot 40}{3.8\times10^6\cdot 0.1} - q_2\,\frac{10}{3.8\times10^6\ 0.1}\right)$$

$$5.26\times10^{-7}\,q_1 - 1.315\times10^{-7}\,q_2$$

$$\theta_2 = \frac{1}{200}\left(-q_1\,\frac{10}{3.8\times10^6\cdot 0.1} + q_2\,\frac{40}{3.8\times10^6\ 0.1}\right)$$

$$-1.316\times10^{-7}\,q_1 + 5.26\times10^{-7}\,q_2$$

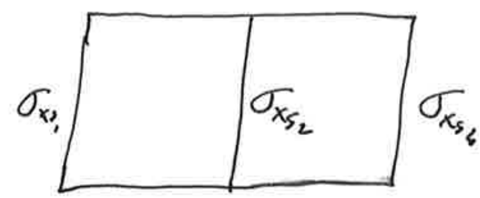
$$\theta_1 - \theta_2 = 6.57\times10^{-7} q_1 - 6.57\times10^{-7} q_2$$

Matrix

$$\begin{bmatrix} 200 & 200 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 10000 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{matrix} 25 \\ 25 \end{matrix}$$

Stress



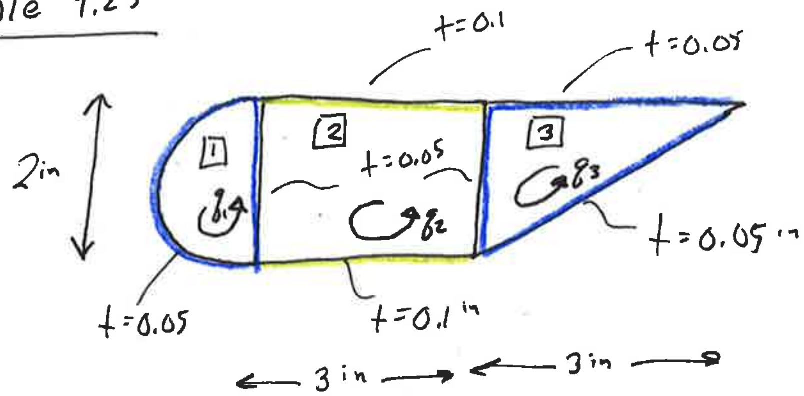
$$\sigma_{xs_1} = \frac{q}{t} = 250 \text{ psi}$$

$$\sigma_{xs_2} = \frac{q_1 - q_2}{t} = 0$$

$$\sigma_{xs_3} = \frac{q}{t} = 250 \text{ psi}$$

Again, the center web is useless

Example 4.25



$M_x = 1000$ lb in

$G = 4.0 \times 10^6$ psi

① Loading $M_x = 1000$ lb in

② Section Properties

$$A_1 = \frac{1}{2}\pi 1^2 = 1.57^{\text{in}^2} \quad A_2 = 3 \cdot 2 = 6^{\text{in}^2} \quad A_3 = \frac{1}{2} \cdot 3 \cdot 2 = 3^{\text{in}^2}$$

③ Applied Torque

$$M_x = \sum 2qA = 2q_1A_1 + 2q_2A_2 + 2q_3A_3$$

$$= 3.1415\, q_1 + 12 q_2 + 6 q_3$$

④ Twist

$$\theta_1 = \frac{1}{2A}\oint \frac{q}{Gt}ds = \frac{1}{3.1415}\left(q_1 \frac{\frac{1}{2}\pi 2 + 2}{4\times10^6 \cdot 0.05} - q_2 \frac{2}{4\times10^6 \cdot 0.05}\right)$$

$$= 8.183 \times 10^{-6} q_1 - 3.18 \times 10^{-6} q_2$$

$$\theta_2 = \frac{1}{12}\oint \frac{q}{Gt}ds = \frac{1}{12}\left(-q_1\left(\frac{2}{4\times10^6 \cdot 0.05}\right) + q_2 \frac{2+2}{4\times10^6\ 0.05} + q_2 \frac{3+3}{4\times10^6 \cdot 0.1} - q_3 \frac{2}{4\times10^6\ 0.05}\right)$$

$$-8.33\times10^{-7} q_1 + 2.916\times10^{-6} q_2 - 8.33\times10^{-7} q_3$$

$$\theta_3 = \frac{1}{6}\oint \frac{q}{Gt}ds = \frac{1}{6}\left(-q_2 \frac{2}{4\times10^6 \cdot 0.05} + q_3 \frac{3 + 2 + \sqrt{3^2+2^2}}{4\times10^6 \cdot 0.05}\right)$$

$$-1.66\times10^{-6} q_2 + 7.17\times10^{-6} q_3$$

$$\theta_1 - \theta_2 = 9.016\times10^{-6} q_1 \quad - 6.096\times10^{-6} q_2 \quad + 8.33\times10^{-7} q_3$$

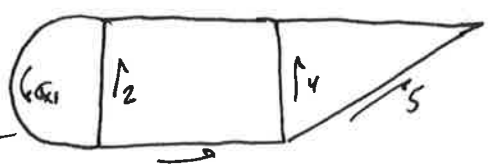
$$\theta_2 - \theta_3 = 8.183\times10^{-6} q_1 \quad - 1.52\times10^{-6} q_2 \quad - 7.17\times10^{-6} q_3$$

(5) Matrix

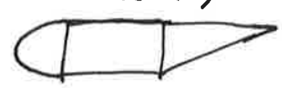
$$\begin{bmatrix} 3.1415 & 12 & 6 \\ 90.2 & -61 & 8.33 \\ 8.18 & -1.52 & -7.17 \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 1000 \\ 0 \\ 0 \end{pmatrix}$$

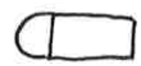
$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 37 \\ 58.7 \\ 29.8 \end{pmatrix}$$

(6) Stress



$\sigma_{xs} = \frac{q_1}{t}$





	this config	Old config
$\sigma_{xs_1} = \frac{q_1}{t} =$	740 psi	958 psi
$\sigma_{xs_2} =$	-435 psi	-458 psi
$\sigma_{xs_3} =$	587 psi	708 psi
$\sigma_{xs_4} =$	~~[illegible]~~ 578 psi	1416 psi
$\sigma_{xs_5} =$	596 psi	N/A

(7)

$\theta = 1.16\times10^{-4}$

$$TR = \frac{M_x}{\theta} = 8.6\times10^{6} > 6.0\times10^{6}$$

What happened?