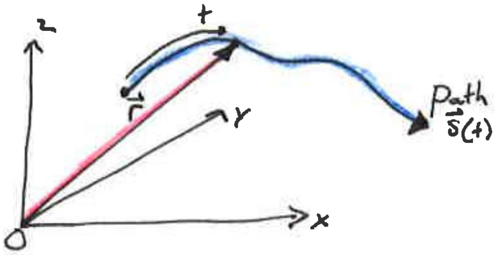
Lesson 8
Equations of Motion

Particle Kinematics



position measured from O "origin" is $\vec{r}(t)$ tracing out a path $s(t)$

The velocity is defined as the change in position with respect to time

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \frac{d\vec{r}}{ds}\dot{s}$$

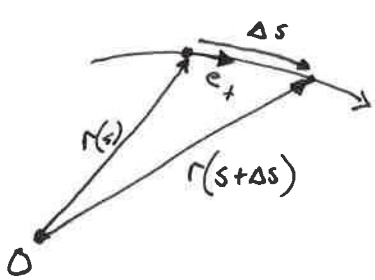
The acceleration is the rate of change of velocity

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{r}}{ds}\ddot{s} + \frac{d^2 r}{ds\,dt}\dot{s} = \frac{d\vec{r}}{ds}\ddot{s} + \frac{d}{dt}\left(\frac{dr}{ds}\right)\dot{s}$$

$$= \frac{d\vec{r}}{ds}\ddot{s} + \frac{d}{ds}\left(\frac{dr}{dt}\right)\dot{s} = \frac{d\vec{r}}{ds}\ddot{s} + \frac{d}{ds}\left(\frac{dr}{ds}\frac{ds}{dt}\right)\dot{s}$$

$$= \frac{d\vec{r}}{ds}\ddot{s} + \frac{d^2\vec{r}}{ds^2}\dot{s}^2$$

What is dr/ds ?



$$\frac{dr}{ds} = \lim_{\Delta s \to 0} \frac{r(s+\Delta s) - r(s)}{\Delta s}$$

$$\equiv e_t$$

The tangent direction vector

$e_t \cdot e_t = 1$ unit vector

$$\boxed{\frac{dr}{ds} = e_t}$$

How does $\frac{dr}{ds}$ change along s? what is $\frac{de_t}{ds}$?

Remember that e_t is a unit vector.

$$e_t \cdot e_t = 1$$

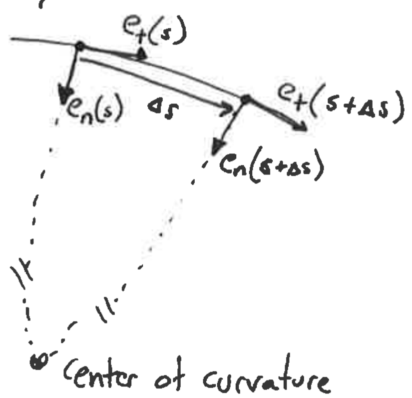
take $\frac{d}{ds}$ derivative.

$$\frac{d}{ds}\left(e_t \cdot e_t\right) = \frac{d}{ds}(1)$$

$$e_t \cdot \frac{de_t}{ds} + \frac{de_t}{ds} \cdot e_t = 0 \quad \Rightarrow \quad \underbrace{e_t \cdot \frac{de_t}{ds}}_{} = 0$$

this says that $\frac{de_t}{ds}$ is perpendicular to e_t.

Visually

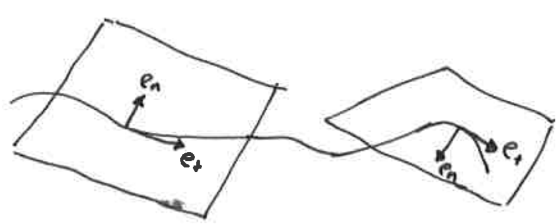


$$\vec{e}_n = \rho \frac{de_t}{ds}$$

$$\rho = \left|\frac{de_t}{ds}\right|^{-1} \quad \text{inverse absolute magnitude.}$$

$= $ radius of curvature

The e_t and e_n vectors form the osculating plane.



The vector orthogonal to both e_t and e_n is the binormal e_b

$$e_b = e_t \times e_n$$

Newton's 2^{nd} Law

$$\vec{V} = V e_t = \dot{s} e_t$$

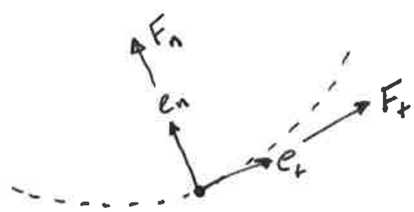
$$\vec{a} = \dot{V} e_t + \frac{V^2}{\rho} e_n$$

Newton Law is $F = ma$ for each component direction

$$\Sigma F_t = \Sigma F \cdot e_t = m\dot{V}$$

$$\Sigma F_n = \Sigma F \cdot e_n = m\frac{V^2}{\rho}$$

$$\Sigma F_b = \Sigma F \cdot e_b = 0$$



Ex:

$e_t = (1, 0)$ and $e_n = (0, 1)$

Given a force $F = (10, 10)^{lbf}$ and $V = 10^{ft/s}$, for a 1 slug object determine the radius of curvature and speed rate of change.

$$F_t = F \cdot e_t = (10, 10) \cdot (1, 0) = 10^{lbf} = m\dot{V} = 1 \text{ slug}$$

$$\dot{V} = \frac{10\,\cancel{lbf}}{1\,\cancel{slug}} \cdot \frac{\cancel{slug}\,ft}{\cancel{lbf}\,s^2} = 10\, ft/s^2$$

$$F_n = F \cdot e_n = (10, 10) \cdot (0, 1) = 10^{lbf} = m\frac{V^2}{\rho}$$

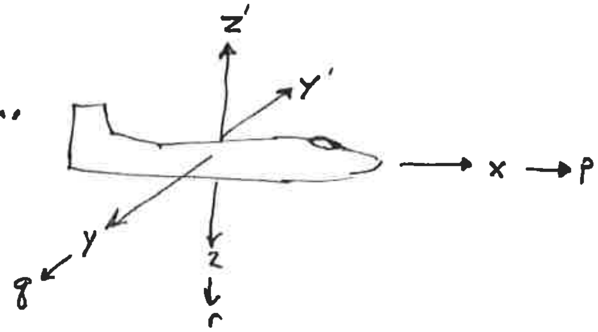
$$\rho = \frac{mV^2}{F_n} = \frac{1\,\cancel{slug}}{} \cdot \frac{10^2\,ft^{\cancel{2}}}{\cancel{s^2}} \cdot \frac{}{10\,\cancel{lbf}} \cdot \frac{\cancel{lbf}\,\cancel{s^2}}{\cancel{slug}\,\cancel{ft}} = 10\, ft$$

$$\boxed{\dot{V} = 10\, ft/s \qquad \rho = 10\, ft}$$

~~Inertial Navigation System (INS)~~

Reference Frames

Body Frame

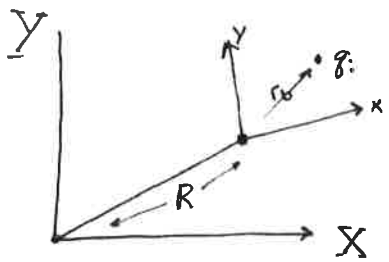


(Non-inertial frame)

Body fixed accelerations + rotations (Strapdown)

x, y, z   p, q, r

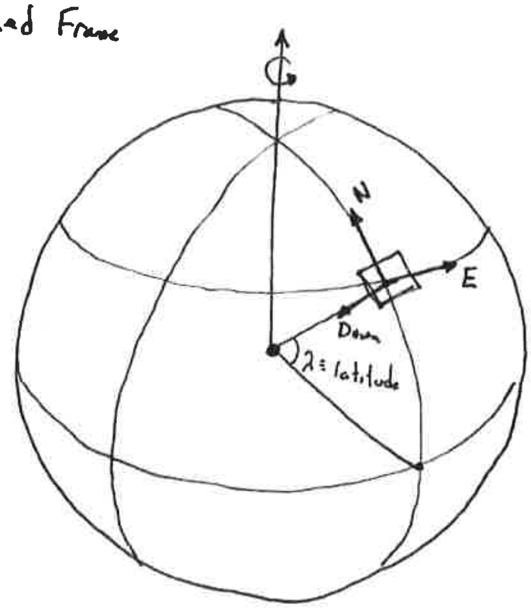
Inertial Frame



$$q_i = R_i + Br_b$$

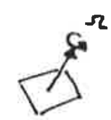
$\uparrow$ Orientation
- Euler Angles
- Direction Cosines
- Quaternions

Earth Fixed Frame



Earth rotates at 15°/hr

$$\Omega_{vertical} = 15 \sin \lambda \left[\frac{deg}{h}\right]$$



$$\Omega_{horizontal} = 15 \cos \lambda$$

- East direction has zero Earth rotation rate. You can find North by finding East with a rate gyro! (Actually easier to find zero rate than maximum rate.)

Realistic Earth Model

The Earth is <u>not</u> an isotropic sphere.

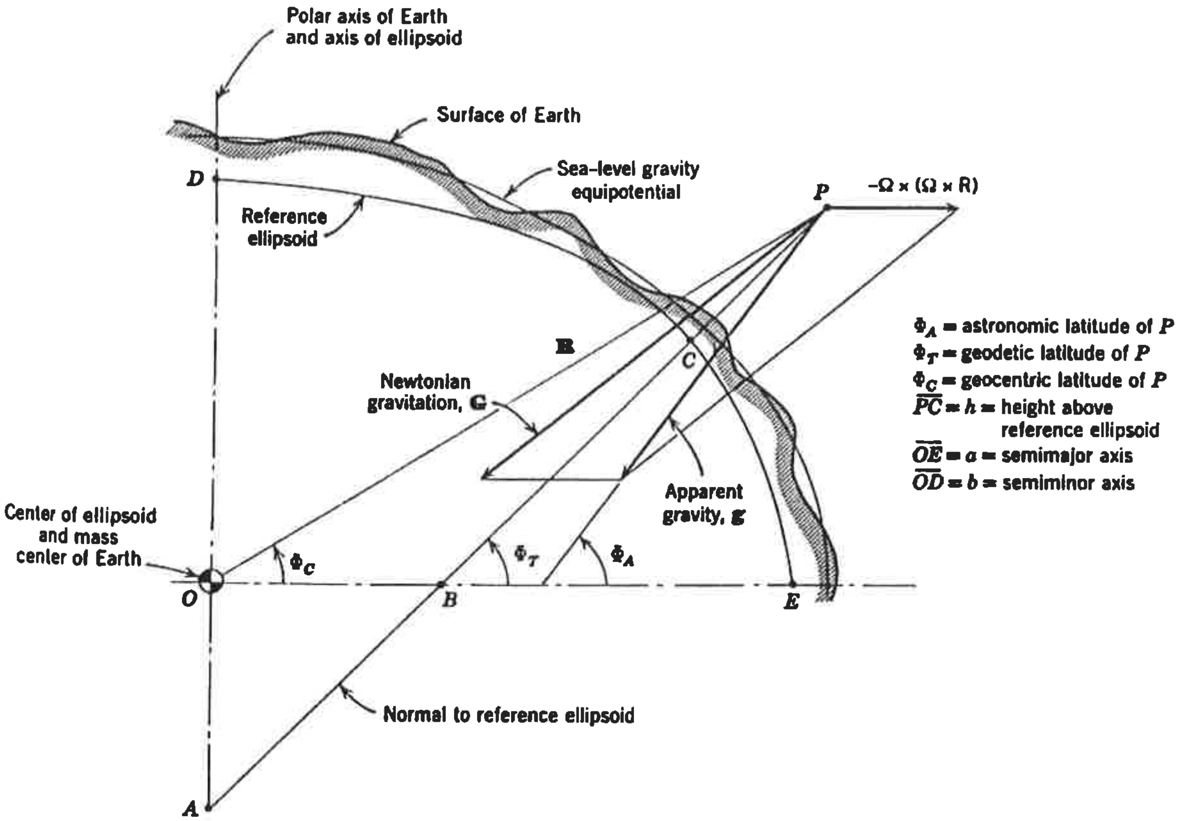
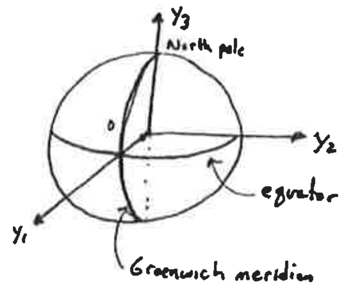


Figure 2.2 Meridian section of the Earth, showing the reference ellipsoid and gravity field.

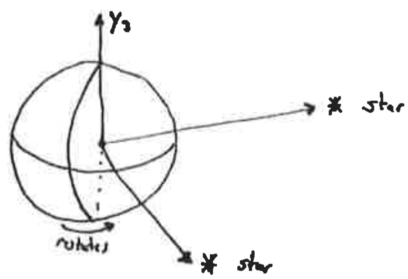
Coordinate Frames

- Earth-centered Earth fixed (ECEF)



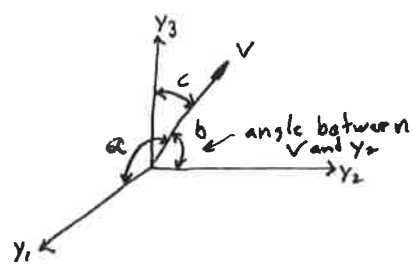
Coordinate system rotates with Earth

- Earth Centered Inertial



Newtons laws valid in this frame.
By definition, good for celestil navigation

- Direction Cosines



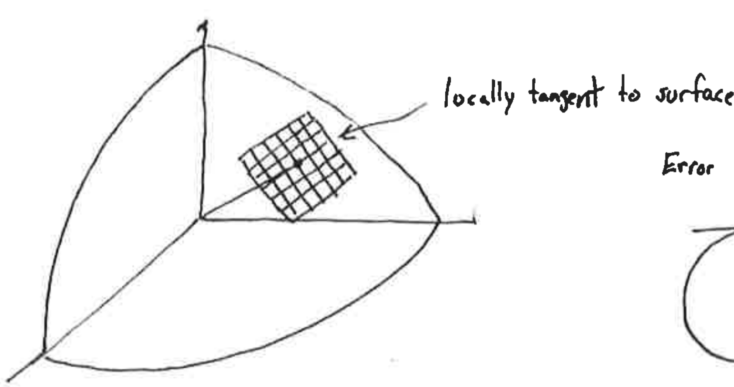
$$V = V_{Y_1}\hat{e}_{Y_1} + V_{Y_2}\hat{e}_{Y_2} + V_{Y_3}\hat{e}_{Y_3}$$

$$\alpha = \cos a = \frac{V \cdot e_{Y_1}}{|V|} = \frac{V_{Y_1}}{(V_{Y_1}^2 + V_{Y_2}^2 + V_{Y_3}^2)}$$

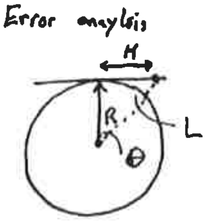
$$\beta = \cos b = \frac{V \cdot e_{Y_2}}{|V|} = \cdots$$

$$\underbrace{\gamma}_{\text{direction cosines}} = \cos \underbrace{c}_{\text{direction angles}} = \frac{V \cdot e_{Y_3}}{V} = \cdots$$

- Tangent plane



Error anaylsis



Radius ≈ 3960 mi ≈ 4000 mi

Along the circle: $L = R\theta$

Along the plane: $\tan\theta = \frac{H}{R}$

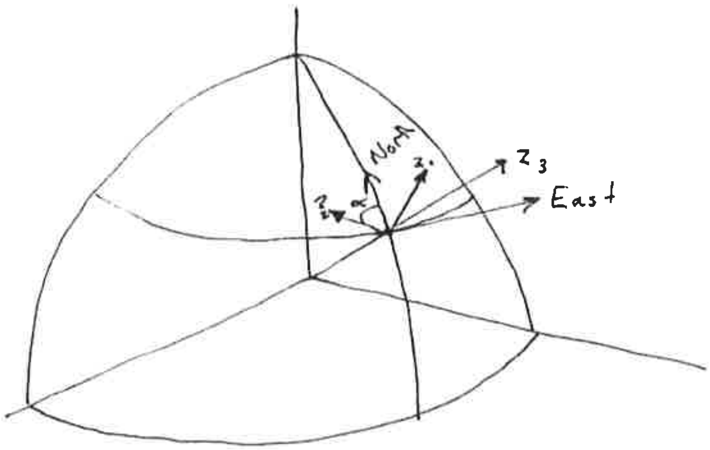
Solve $\tan\frac{L}{R} = \frac{H}{R}$

$$\tan\frac{L}{R} \approx \frac{L}{R} + \frac{1}{3}\left(\frac{L}{R}\right)^3 + \frac{2}{15}\left(\frac{L}{R}\right)^5 + \cdots$$

$$\text{Error} = \tan\frac{L}{R} - \frac{H}{R}$$

$\approx$

- Geodetic Wander Azimuth



Z_1 and Z_2 in tangent plane

Z_3 upward

Z_2 angled α from north.

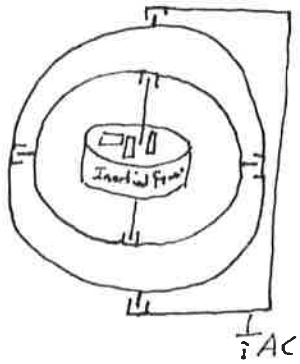
- Others

...

Two types of INS platforms.

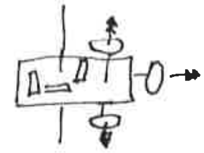
1) Inertial Gyroscopic platform

- Passive

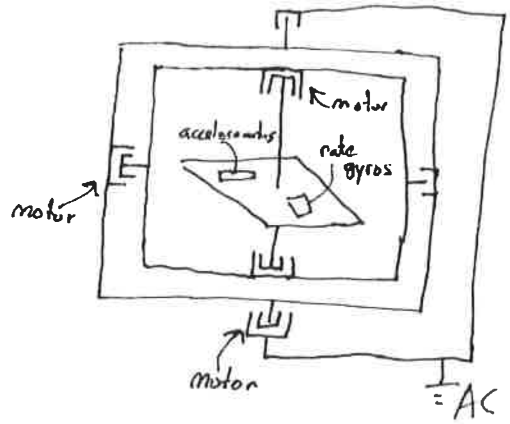


Issues?
Drift, power, Bearing, stiction.

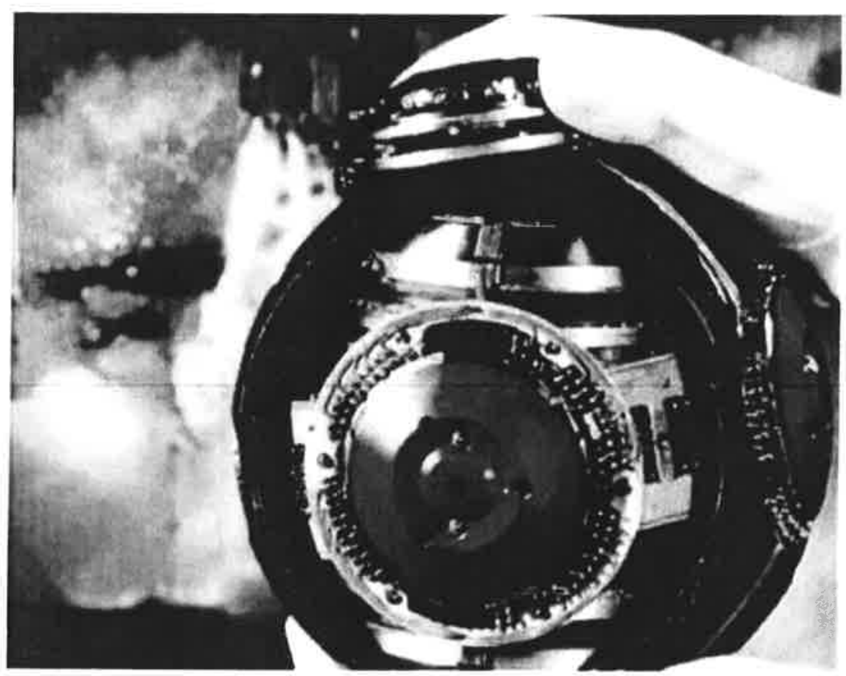
Alternative platform with gyros



- Active



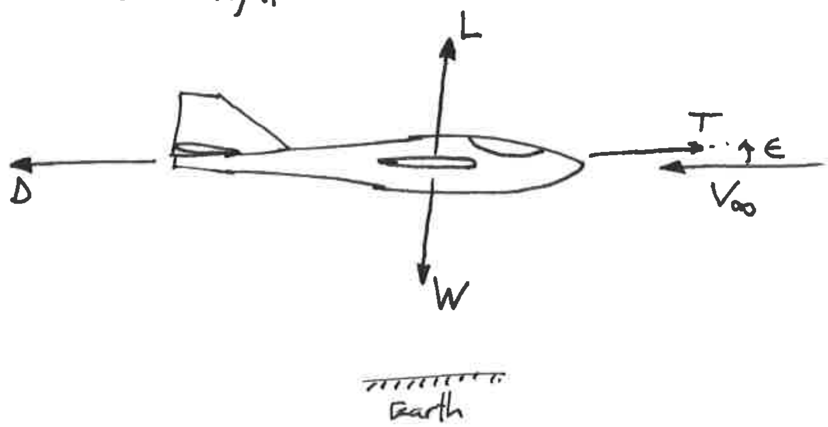
- Gyros are coupled with motors on the gimbals to create a feedback mechanism to ensure the platform remains in an inertial frame.
- Gyros are "null seeking" to drive error to zero. "Integrating gyros" means they track the integral of ω.



Source:

Inertial Navigation – Forty Years of Evolution A. D. King GEC ... 1998

Four Forces of Flight



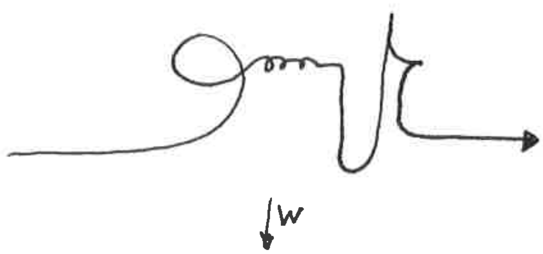
$L \perp V_\infty$

$D \parallel V_\infty$

W towards Earth's center

T at ϵ to V_∞

Earth fixed frame or Body fixed frame?



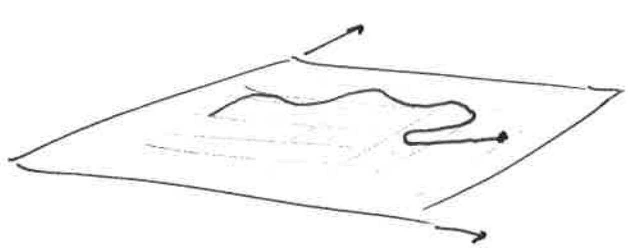
Which direction is $\underbrace{L, D, T}_{\text{body frame}}, \underbrace{W}_{\text{global frame}}$?

We will go into details of orientation angles later in the semester.

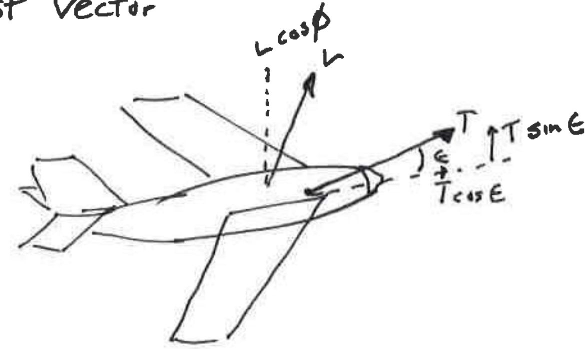
For now, roll angle $\emptyset$

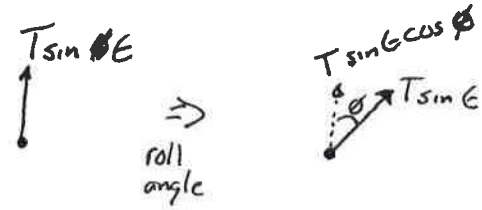
pitch angle $\emptyset$

For now, we will also assume a flat earth.



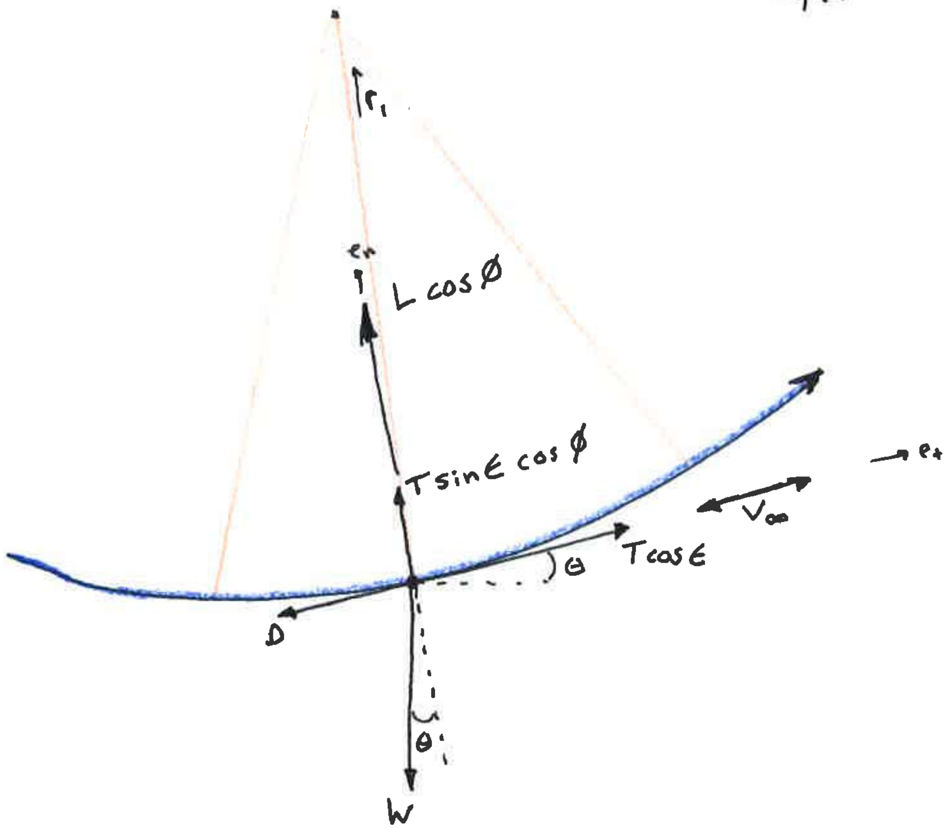
Thrust vector





Thrust is projected into z direction and roll angle.

Forces



In e_t direction

$$m\frac{dV_\infty}{dt} = T\cos\epsilon - D - W\sin\theta$$

In e_n direction.

$$m\frac{V_\infty^2}{r_1} = L\cos\phi + T\sin\epsilon\cos\phi - W\cos\theta$$

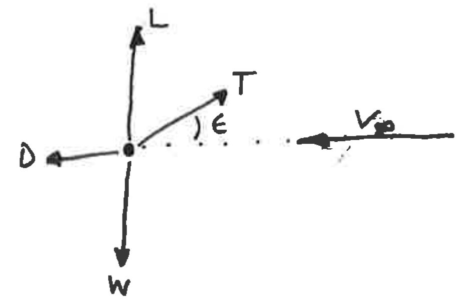
Lesson 8 part 2

Level Flight

Steady Level Flight

$a = \frac{dv}{dt} = 0$ (Steady)

no climb angle, no bank angle (Level)



Horizontal Direction

$$-D + T\cos\epsilon = 0$$

Vertical Direction

$$-W + L + T\sin\epsilon = 0$$

For small thrust angles, $\epsilon \approx 0 \Rightarrow \sin\epsilon \approx 0$ and $\cos\epsilon \approx 1$

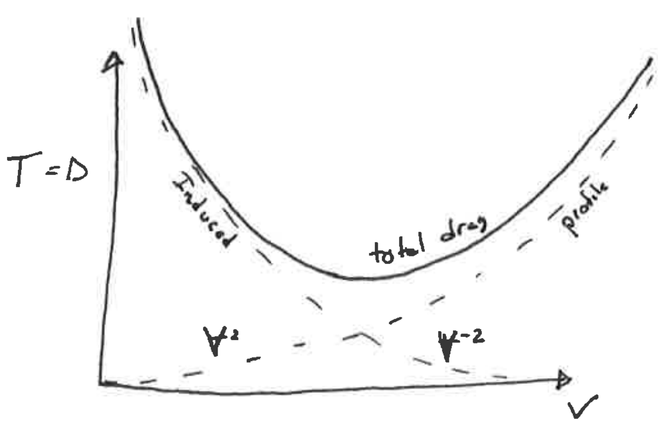
$$-D + T = 0 \Rightarrow \boxed{T = D}$$

$$-W + L + 0 = 0 \Rightarrow \boxed{L = W}$$

Given an aircraft with $C_D = C_{D_0} + kC_L^2$ and $C_L = \frac{L}{\frac{1}{2}\rho V^2 S}$

$$L = W = \frac{1}{2}\rho V^2 S C_L \Rightarrow C_L = \frac{2W}{\rho V^2 S}$$

$$D = T = \frac{1}{2}\rho V^2 S\left(C_{D_0} + kC_L^2\right) \Rightarrow T = \frac{1}{2}\rho V^2 S\left(C_{D_0} + k\frac{4W^2}{\rho^2 V^4 S^2}\right)$$



$$= \underbrace{\frac{1}{2}\rho V^2 S C_{D_0}}_{V^2} + \underbrace{\frac{2W^2 K}{\rho V^2 S}}_{V^{-2}}$$

The point of minimum drag is where the induced drag equals profile drag.

Minimum Thrust

Find $\frac{dT}{dV} = 0 = \rho V S C_{D_0} - \frac{4 W^2 K}{\rho V^3 S}$

$$V = \sqrt[4]{\frac{K}{C_{D_0}}} \sqrt{\frac{2W}{\rho S}}$$

for most aircraft

$k = \frac{1}{\pi A R e} \gtrsim C_{D_0} \approx 200$ counts or so

Thus $\sqrt[4]{\frac{K}{C_{D_0}}}$ is somewhat flat

$\frac{W}{S}$ is wing loading

subst'

$$T = \frac{1}{\cancel{2}} \cancel{\rho} \sqrt{\frac{K}{C_{D_0}}} \frac{\cancel{2}W}{\cancel{\rho S}} \cancel{S} C_{D_0} + \frac{\cancel{2}W^{\cancel{2}}K}{\cancel{\rho S}} \sqrt{\frac{C_{D_0}}{K}} \frac{\cancel{\rho S}}{\cancel{2W}}$$

$$= \underbrace{\sqrt{\frac{K}{C_{D_0}}} W C_{D_0}}_{} + \underbrace{\sqrt{K C_{D_0}} W}_{} = \boxed{2\sqrt{K C_{D_0}}\, W = T_{min}}$$

$=$

or $\boxed{\frac{T}{W} = 2\sqrt{K C_{D_0}}}$

Or since $L = W$ and $T = D$

$$\boxed{\left(\frac{L}{D}\right)_{max} = \frac{1}{2\sqrt{K C_{D_0}}}}$$

With our aero model

$$C_{D_i} = K C_L^2 \approx \frac{C_L^2}{\pi A R e} \Rightarrow k = \frac{1}{\pi A R e}$$

$$\left(\frac{L}{D}\right)_{min} = \frac{1}{2\sqrt{\frac{C_{D_0}}{\pi A R e}}}$$

$$\boxed{\left(\frac{L}{D}\right)_{max} = \frac{1}{2}\sqrt{\frac{\pi A R e}{C_{D_0}}}}$$

larger AR → higher $\frac{L}{D}$

lower AR → lower L/D

Ex:

For an aircraft with $C_{D_0} = 0.0200$ and $AR = 10$ and $e \approx 0.8$, what is the largest L/D ratio possible?

$$\left(\frac{L}{D}\right)_{max} = \frac{1}{2}\sqrt{\frac{\pi \cdot 10 \cdot 0.8}{0.0200}} = 17.7$$

at what speed? $W/S = 10 \frac{lbf}{ft^2}$, $\rho = 0.00237 \frac{slug}{ft^3}$

$$V = \sqrt[4]{\frac{K}{C_{D_0}}}\sqrt{\frac{2}{\rho}\frac{W}{S}} \qquad K = \frac{1}{\pi ARe}$$

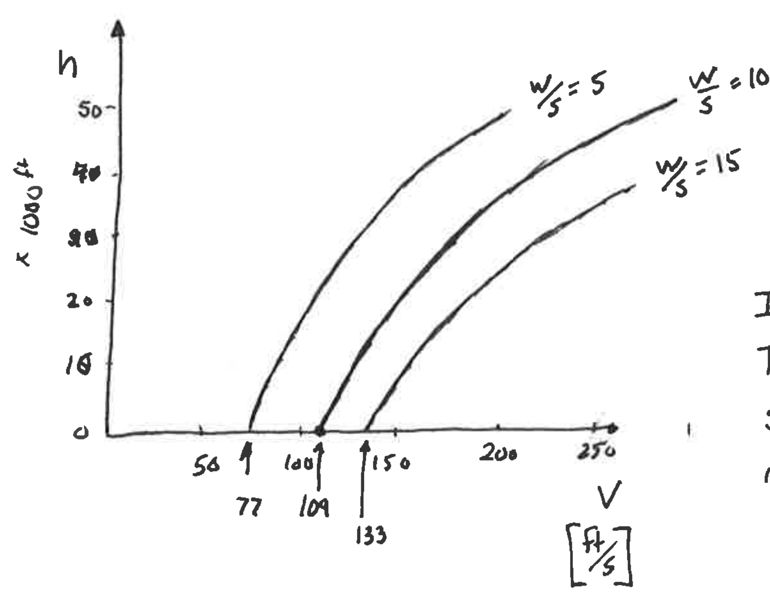
$$= \sqrt[4]{\frac{1}{\pi ARe C_{D_0}}}\sqrt{\frac{W}{S}}\sqrt{\frac{2}{\rho}}$$

$$= \sqrt[4]{\frac{1}{\pi \cdot 10 \cdot 0.8 \cdot 0.02}}\sqrt{10 \frac{lbf}{ft^2}}\sqrt{\frac{2 \cdot ft^3}{0.00237\ slug} \left| \frac{slug\ ft\ ft}{lbf\ s^2}\right.}$$

$$= 1.18 \quad \cdot \quad 91.86 \frac{ft}{s}$$

$$\boxed{V_{min\ thrust} = 109 \frac{ft}{s}}$$

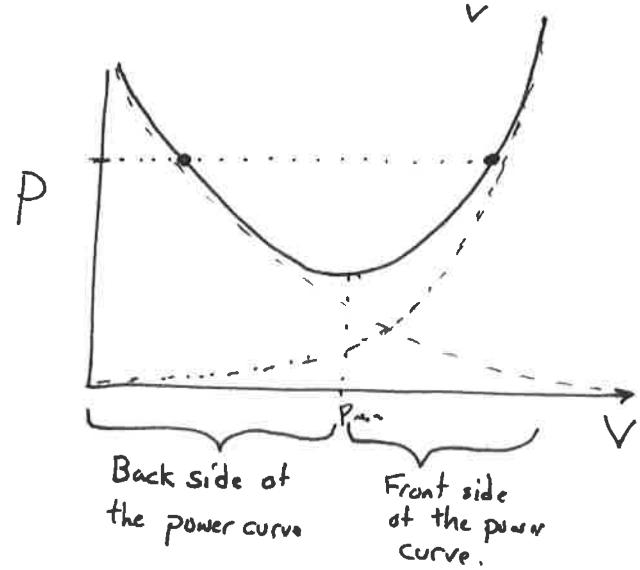
How does this change with altitude?



If fuel burn depends on TSFC, at what altitude should you operate at to maximize range?

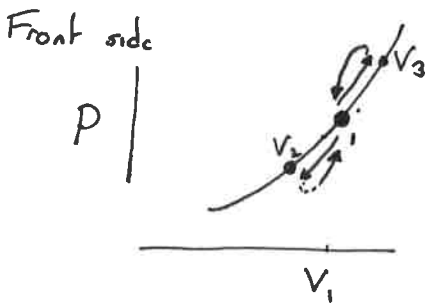
Power

$$P = T \cdot V = \underbrace{\frac{1}{2}\rho V^3 S C_{D_0}}_{V^3} + \underbrace{\frac{2W^2K}{\rho V S}}_{\frac{1}{V}}$$



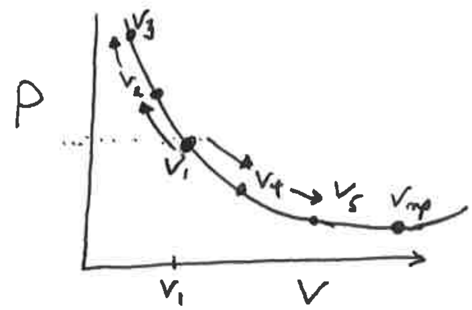
For the same power setting, the aircraft has two SLF velocities.

Evaluate the speed stability for a given power setting.



- operating at velocity V_1
- Slow down slightly to V_2
  Result: less power to fly SLF!
  Thus extra power tends to accelerate the aircraft back to V_1
- Speed up slightly to V_3
  Result: More power required to fly SLF
  A/C decelerates back to V_1

Back side of Power curve



- Operating at velocity V_1
- Slow down slightly to V_2
  Result: More power to fly SLF
  A/C decelerates further to V_3
  A/C would eventually stall
- Speed up slightly to V_4
  Result: Less Power to fly SLF
  A/C accelerates to V_5
  A/C approaches minimum power V_{mp}

SLF on the back side (less than $V_{min\ power}$) is more challengin.
The aircraft will not automatically maintain a speed given an altitude hold.

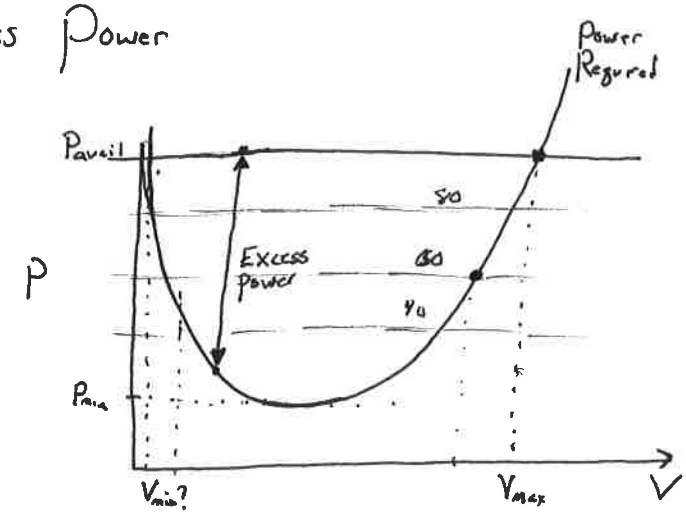
Minimum Power.

Find
$$\frac{dP}{dV} = 0 = \frac{3}{2}\rho V^2 S C_{D_0} + - \frac{2W^2K}{\rho V^2 S}$$

solve for V

$$V = \sqrt[4]{\frac{4}{3}\frac{W^2K}{S^2 C_{D_0} \rho}}$$

Excess Power



Where does this excess power go?

- Increase height (climb) $W \cdot V_{vertical}$
- Increase speed (accelerate) $m \frac{dV_\infty}{dt} > 0$

Maximum/Minimum Speed

When $P_{avail} = P_{req}$, the SLF top/maximum speed is reached. (front side of curve)

Is the minimum speed when $P_{avail} = P_{req}$ (on back side)?

No. You may reach $C_{L_{max}}$ prior to the power limit

This is common in high performance aircraft (WW2)