# Lesson 9 part 2
# Rate of Climb

# Definitions:

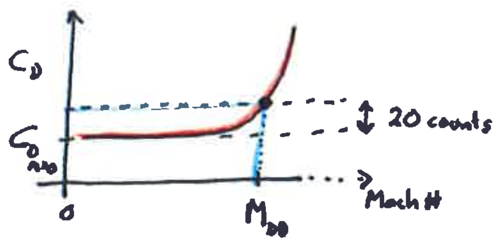
**Critical Mach #:** The Mach number where $M=1$ first exists in the flow field.

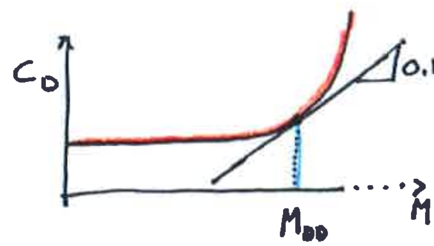
$M_{crit}$


The critical Mach number for an NACA 0012 is M 0.77.

**Drag Divergence Mach #:** (multiple definitions) $M_{DD}$

1) When $\Delta C_D = 20$ counts from the subsonic value — count $\equiv 0.0001$
2) When $\left.\dfrac{dC_D}{dM}\right|_{C_L \text{ constant}} = 0.1$



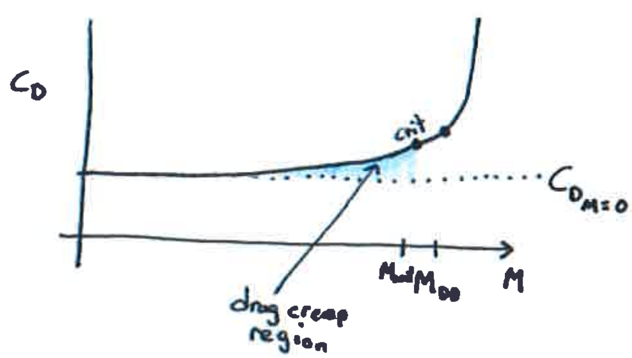




$$M_{DD_{20\text{ counts}}} \neq M_{DD_{0.1\text{ slope}}}$$

I prefer this one!

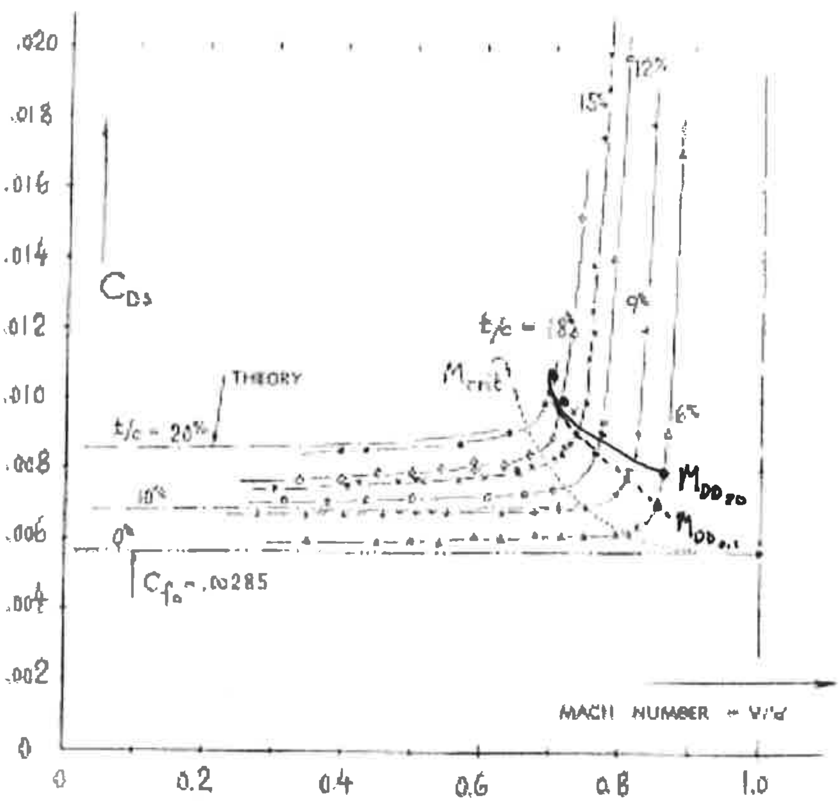
**Drag Creep:**

Often, the drag slowly begins increasing <u>below</u> $M_{crit}$.




A rough or thick airfoil (of an older design especially) tends to show this behavior. This also shows up for entire aircraft for similar reasons.

# Fluid Dynamic Drag    Hoerner



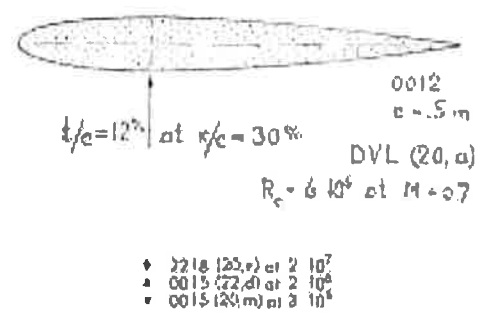
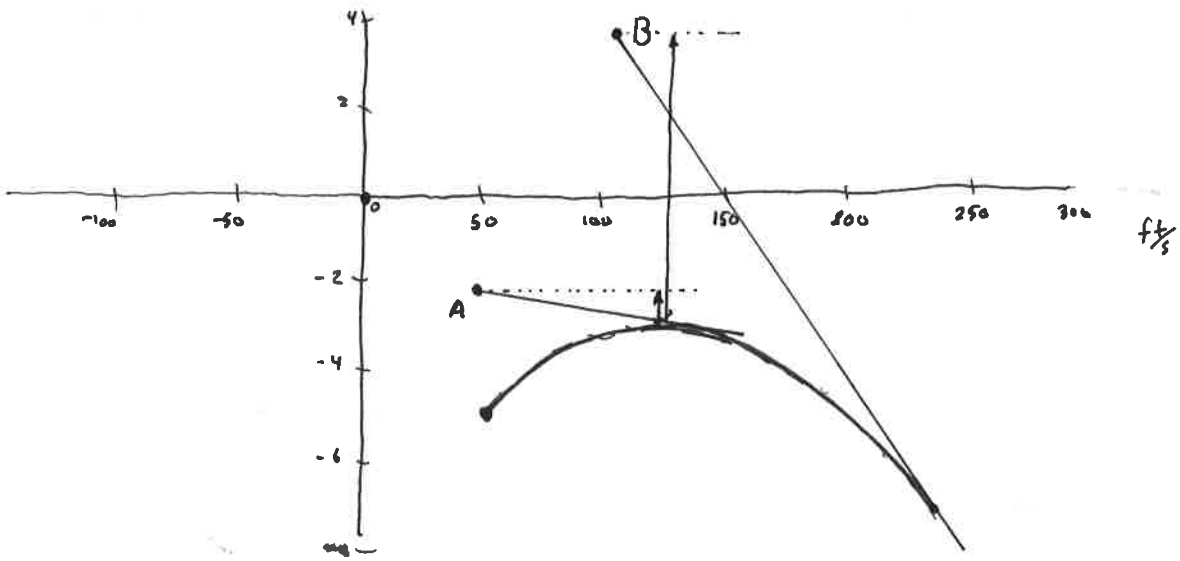




Figure 13. Drag coefficients (at $C_L = 0$) of a *family of symmetrical foil sections* (with maximum thickness at 30% of the chord); (a) tested in a large-size wind-tunnel (20,a); and (b) calculated as per equation 18 for $C_f = 0.00285$ = constant.

0015 at transonic speeds, see (36,a) in Chapter XVII

# Hodograph in Wind (Corrected)

Shift the ~~starting~~ origin point for wind and sink.




**A**: 50 ft/s headwind and 2 ft/s updraft

$\dot{h}_{min} \approx -0.7$ ft/s @ 120 ft/s

$\theta_{min} \approx -0.5°$ @ 140 ft/s $\qquad \theta \approx \operatorname{atan}\left(\frac{-0.8 \text{ ft/s}}{140-50}\right) \approx -0.5°$

B: 100 ft/s headwind and 4 ft/s downdraft

$\dot{h}_{min} \approx -6.7$ ft/s @ 120 ft/s

$\theta_{min} \approx -5°$ @ 230 ft/s $\qquad \theta \approx \operatorname{atan}\left(\frac{-11.0 \text{ ft/s}}{230-100 \text{ ft/s}}\right) \approx -5°$

B: Suboptimal

If we had picked to fly @ 50 ft/s airspeed,

$\dot{h} \approx -9$ ft/s

$\theta \approx \operatorname{atan}\left(\frac{-9}{50-100}\right) \approx 10°$ backwards

At an airspeed of 100 ft/s

$\dot{h} \approx -7$ ft/s

$\theta = 90°$

down

# Maximum Rate of Climb

previously, we found: $\dot{h} = V_\infty \Big( \frac{T}{W} - \frac{1}{2}\rho V_\infty^2 \left(\frac{W}{S}\right)^{-1} C_{D_0} - \underbrace{\left(\frac{W}{S}\right) \frac{2k\cos^2\theta}{\rho V_\infty^2}}_{\text{for most practical problems, } \cos^2\theta \approx 1}$

Assuming $\theta$ is relatively small,

$$\dot{h} = V_\infty \Big( \frac{T}{W} - \frac{1}{2}\rho V_\infty^2 \left(\frac{W}{S}\right)^{-1} C_{D_0} - \left(\frac{W}{S}\right) \frac{2k}{\rho V_\infty^2}$$

Find the maximum $\dot{h}$ wrt $V_\infty$

$$\frac{d\dot{h}}{dV_\infty} = \frac{T}{W} - \frac{1}{2} 3\rho V_\infty^2 \left(\frac{W}{S}\right)^{-1} C_{D_0} + \left(\frac{W}{S}\right) \frac{2k}{\rho V_\infty^2} = 0$$

Divide by $-\frac{3}{2}\rho_\infty \left(\frac{W}{S}\right)^{-1} C_{D_0}$ to isolate the $V_\infty^2$ in the middle term

$$V_\infty^2 - \frac{2}{3}\left(\frac{T}{W}\right)\left(\frac{W}{S}\right)\frac{1}{\rho C_{D_0}} - \frac{4}{3}\left(\frac{W}{S}\right)\left(\frac{W}{S}\right)\frac{k}{\rho^2 V_\infty^2}\frac{1}{C_{D_0}}$$

Mult by $V_\infty^2$

$$V_\infty^4 - \frac{2}{3}\left(\frac{T}{W}\right)\left(\frac{W}{S}\right)\frac{1}{\rho C_{D_0}} V_\infty^2 - \frac{4}{3}\left(\frac{W}{S}\right)^2 \frac{k}{C_{D_0}}\frac{1}{\rho^2} = 0$$

This is quadratic in $V_\infty^2$ $\Big( \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$V_\infty^2 = \frac{1}{3}\frac{T}{W}\frac{W}{S}\frac{1}{\rho C_{D_0}} + \frac{1}{\rho}\frac{W}{S}\sqrt{\frac{1}{9}\left(\frac{T}{W}\right)^2 \frac{1}{C_{D_0}^2} + \frac{4}{3}\frac{k}{C_{D_0}}}$$

pull $\frac{T}{W}$ out of $\sqrt{\ }$ by mult and div by $\left(\frac{T}{W}\right)^2$

$$V_\infty^2 = \frac{1}{3}\left(\frac{T}{W}\right)\left(\frac{W}{S}\right)\frac{1}{\rho C_{D_0}} + \frac{1}{\rho}\left(\frac{W}{S}\right)\left(\frac{T}{W}\right)\sqrt{\frac{1}{9}\frac{1}{C_{D_0}^2} + \frac{4}{3}\frac{k}{C_{D_0}}\frac{1}{(T/W)^2}}$$

Remember that $\left(\frac{L}{D}\right)_{max} = \frac{1}{\sqrt{4kC_{D_0}}}$ substitute into k

$$V_\infty^2 = \frac{1}{3}\left(\frac{T}{W}\right)\left(\frac{W}{S}\right)\frac{1}{\rho C_{D_0}} \cdot \left( 1 + \sqrt{1 + \frac{3}{\left(\frac{L}{D}\right)_{max}^2 \left(\frac{T}{W}\right)^2}} \right)$$

$$V = \sqrt{\frac{1}{3}\left(\frac{T}{W}\right)\left(\frac{W}{S}\right)\frac{1}{\rho C_{D_0}} \cdot \left(1 + \sqrt{1 + \frac{3}{\left(\frac{L}{D}\right)_{max}^2 \left(\frac{T}{W}\right)^2}}\right)}$$

You may notice in the book that this is for a <u>jet aircraft</u>.

Why? We made a hidden assumption. $T$ is constant wrt $V_\infty$.

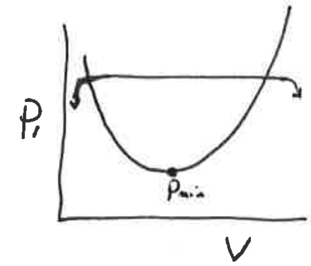
Such that $\frac{d\dot{h}}{dV} = \frac{d}{dV}\left(V_\infty \frac{T}{W}\right) + \cdots = \frac{T}{W}$

The rate of climb corresponding to this velocity is found by substituting into $\dot{h}$. (Messy...)

$$\dot{h} = \left[\frac{1}{3}\frac{1}{\rho C_{D_0}}\frac{W}{S}\underbrace{\left(1 + \sqrt{1 + \frac{3}{\left(\frac{L}{D}\right)_{max}^2\left(\frac{T}{W}\right)^2}}\right)}_{\text{“}Z\text{”}}\right]^{1/2}\left(\frac{T}{W}\right)^{3/2} \cdot \left(1 - \frac{Z}{6} - \frac{3}{2}\frac{1}{Z}\left(\frac{T}{W}\right)^{-2}\left(\frac{L}{D}\right)_{max}^{-2}\right)$$

- Strongest term is $\left(\frac{T}{W}\right)^{3/2}$
- $\sqrt{\frac{W}{S}}$
- $\sqrt{\frac{1}{\rho}}$ and $\sqrt{\frac{1}{C_{D_0}}}$

For a propeller aircraft, $T = \frac{P_A}{V_\infty} = \frac{\eta P}{V_\infty}$




Since the power available is relatively constant, the maximum climb rate must occur at the aircraft's minimum power required.

$$P_{req} = V_\infty D_{rag} = V_\infty \frac{1}{2}\rho V_\infty^2 S \left(C_{D_0} + \frac{K C_L^2}{\pi A R e}\right) \quad \text{and} \quad C_L = \frac{2W}{\rho V^2 S} = \frac{2}{\rho V_\infty^2}\left(\frac{W}{S}\right)$$

Minimize wrt $V_\infty$

$$\frac{dP_{req}}{dV_\infty} = \frac{d}{dV_\infty}\left(\frac{1}{2}\rho V_\infty^3 S C_{D_0} + \frac{1}{2}\rho V_\infty^3 S K \left(\frac{2}{\rho V_\infty^2}\frac{W}{S}\right)^2 \frac{1}{\pi A R e}\right)$$

$$= \frac{d}{dV_\infty}\left(\frac{1}{2}\rho V_\infty^3 S C_{D_0} + \frac{1}{2}\rho V_\infty^3 \frac{S K}{\pi A R e} \frac{4}{\rho^2 V_\infty^4}\left(\frac{W}{S}\right)^2\right)$$

$$= \frac{3}{2}\rho V_\infty^2 S C_{D_0} - 2 \frac{S K}{\pi A R e}\left(\frac{W}{S}\right)^2 \frac{1}{V_\infty^2}\frac{1}{\rho} = 0$$

Solve for $V_\infty$

$$\frac{3}{2}\rho V_\infty^4 S C_{D_0} = \frac{2 S K}{\pi A R e}\left(\frac{W}{S}\right)^2 \frac{1}{\rho}$$

$$V_\infty^4 = \frac{K}{C_{D_0}} \frac{1}{\pi A R e} \frac{4}{3} \frac{1}{\rho^2}\left(\frac{W}{S}\right)^2$$

$$V_{\dot{h}_{max}} = \sqrt{\frac{W}{S}\frac{2}{\rho}\sqrt{\frac{K}{3C_{D_0}}\frac{1}{\pi A R e}}} = \sqrt{\frac{W}{S}\frac{2}{\rho}\sqrt{\frac{K}{3C_{D_0}}}}$$

This also is the Velocity that maximizes $\left(\frac{C_L^{3/2}}{C_D}\right)$

plug into $\dot{h}$

$$\dot{h} = V_\infty\left(\frac{T}{W} - \frac{1}{2}\rho V_\infty^2 \left(\frac{W}{S}\right)^{-1} C_{D_0} - \left(\frac{W}{S}\right) \frac{2k\cos^2\theta}{\rho V_\infty^2}\right)$$

$$= \frac{V_\infty \eta P}{W V_\infty} - \frac{1}{2}\rho V_\infty^3 \left(\frac{W}{S}\right)^{-1} C_{D_0} - \left(\frac{W}{S}\right)\frac{2k\cos^2\theta}{\rho V_\infty}$$

$$= \frac{\eta P}{W} - \sqrt{\frac{W}{S}\frac{2}{\rho}\sqrt{\frac{K}{3C_{D_0}}}}\left[\left(\frac{1}{2}\rho \frac{W}{S}\frac{2}{\rho}\sqrt{\frac{K}{C_{D_0}3}}\right)\left(\frac{W}{S}\right)^{-1} C_{D_0} + \frac{W}{S}\sqrt{\frac{3C_{D_0}}{K}}\ \frac{S}{W}\frac{\rho}{2}\frac{2k}{\rho}\right]$$

Simplify

$$= \frac{\eta P}{W} - \sqrt{\frac{W}{S}\frac{2}{\rho}\sqrt{\frac{K}{3C_{D_0}}}}\ \underbrace{\left(\sqrt{\frac{KC_{D_0}}{3}} + \sqrt{3C_{D_0}K}\right)}_{\sqrt{KC_{D_0}}\left(\sqrt{\frac{1}{3}}+\sqrt{3}\right) = \frac{\sqrt{4KC_{D_0}}}{2}\left(\sqrt{\frac{1}{3}}+\sqrt{3}\right) = \frac{1.155}{(L/D)_{max}}}$$

So for a prop/piston plane,

$$\dot{h}_{max} = \underbrace{\frac{\eta P}{W}}_{\text{engine}} - \underbrace{\sqrt{\frac{W}{S}\frac{2}{\rho}\sqrt{\frac{K}{3C_{D_0}}}}\cdot\frac{1.155}{(L/D)_{max}}}_{\text{Aero}} = \frac{\eta P}{W} - \sqrt{\frac{W}{S}\frac{1}{\rho}\frac{1}{C_{D_0}}}\ \frac{0.877}{\left(\frac{L}{D}\right)_{max}^{3/2}}$$

- High $P/W$ ratio
- Low $W/S$
- High $(L/D)_{max}$
- High $C_{D_0}$ (!!?)

# Service and Absolute Ceiling (How high can an aircraft fly?)

$$\text{Climb rate} \equiv \dot{h} = \underbrace{\frac{\text{Excess power}}{\text{Weight}}}_{\substack{\text{The aircraft's absolute ceiling occurs}\\ \text{when excess power is zero}}} = \underbrace{V_\infty\left(\frac{T}{W} - \frac{1}{2}\rho V_\infty^2\left(\frac{W}{S}\right)^{-1} C_{D_0} - \left(\frac{W}{S}\right)\frac{2k\cos^2\theta}{\rho V_\infty^2}\right)}$$

to increase the ceiling,

- Increase $T/W$
- Reduce induced drag + profile drag

Absolute ceiling $\equiv$ zero climb rate on a standard day

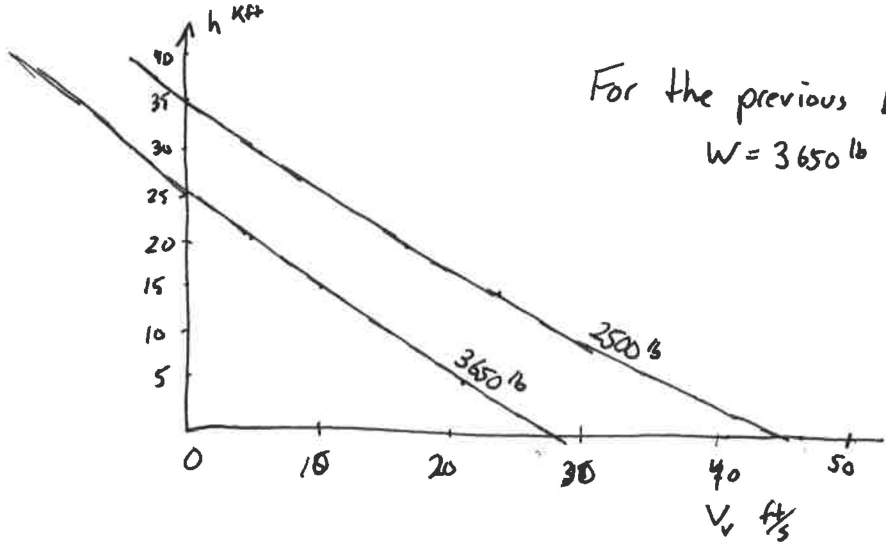
Service ceiling $\equiv$ 50 fpm for small part 23 aircraft
100 fpm for larger aircraft

Maximum Certified Altitude $\equiv$ Legal limit

For a prop/piston aircraft

$$\dot{h} = \frac{\eta P_0 \frac{\rho}{\rho_0}}{W} - \sqrt{\frac{W}{S}\frac{1}{\rho}\frac{1}{C_{D_0}}}\;\frac{0.877}{\left(\frac{L}{D}\right)_{max}^{3/2}} \qquad \text{and} \quad \left(\frac{L}{D}\right)_{max} = \sqrt{\frac{1}{4 C_{D_0} k}}$$

For most aircraft, ~~this~~ $\dot{h}$ is relatively linear with altitude



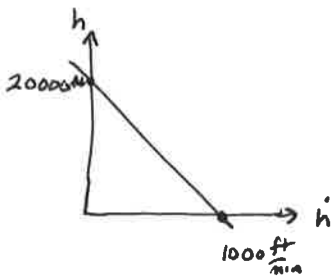

For the previous Bonanza
$W = 3650^{lb}$ $P = 300^{hp}$

This is also totally bogus and wrong, why? $V_{\dot{h}_{max}}(3650^{lbs}, 25^{kft}) = 260^{kts}$ ! $\eta$ would be lower

# Time to Climb

$$\frac{dh}{dt} = \dot{h}(\cdots) \Rightarrow dt = \frac{dh}{\dot{h}} \Rightarrow t = \int_{h_1}^{h_2} \frac{dh}{\dot{h}(\cdots)}$$

Ex: For a prop aircraft, you find $\dot{h}_{SSL} = 1000 \frac{ft}{min}$ and the absolute ceiling is 20000 ft. Estimate the time to climb from SSL to $5^{kft}$, $10^{kft}$, $15^{kft}$, and $20^{kft}$. Assume $\dot{h}$ is linear wrt $h$.




$$\dot{h} \approx 1000\,fpm - 1000 \cdot \frac{h}{20000} = 1000 - \frac{h}{20}$$

$$t_{5kft} = \int_0^{5000} \frac{dh}{1000 - \frac{h}{20}} = 5.7 \text{ min} \qquad \dot{h}_{avg} = 870 \text{ fpm}$$

$$t_{10kft} = \int_0^{10000} \frac{dh}{1000 - \frac{h}{20}} = 13.86 \text{ min} \qquad \dot{h}_{avg} = 720 \text{ fpm}$$

$$t_{15kft} = 27.7 \text{ min} \qquad \dot{h}_{avg} = 540 \text{ fpm}$$

$$t_{20ft} = \infty$$

In general,

$$t = \int_0^{h_f} \frac{dh}{a + bh} = \frac{1}{b}\left(\ln(a + bh_1) - \ln a\right)$$