Lesson 10
Range and Endurance

Range Derivation

Total Weight $\equiv W_{aircraft} + W_{\underset{full}{fuel}}$

Total Weight $= W_{aircraft} + W_{\underset{empty}{fuel}}$

Fuel burn

$$\frac{dW}{dt} = \frac{d}{dt}\left(W_{aircraft} + W_{fuel}\right) = \frac{d}{dt} W_{fuel}$$

Notation

$W \equiv$ total weight

W_f = fuel weight

W_a = empty aircraft weight (dry... no fuel)

Propeller Aircraft

$\underbrace{\text{Specific fuel consumption}}$ is relatively constant with V and h.

$$SFC \equiv \left[\frac{lb}{hp\ h}\right] \qquad c \equiv \left[\frac{lb}{\frac{ft \cdot lbf}{s} \cdot s}\right] = -\frac{\dot{W}_f}{P}$$

$$c_t = \frac{c V_\infty}{\eta} \qquad P = T V_\infty$$

(Converts from specific fuel consumption to thrust specific fuel consumption)

Incremental range

$$ds = V\,dt$$

Fuel burn

$$c = -\frac{\dot{W}_f}{P} \quad \text{or} \quad c_t = -\frac{\dot{W}_f}{T} = -\frac{1}{T}\frac{dW_f}{dt} \Rightarrow dt = -\frac{1}{c_t T} dW_f$$

Substitute

$$ds = -V_\infty \frac{1}{c_t T} dW_f = -V_\infty \frac{\eta}{c V_\infty T} dW_f = \overbrace{-\frac{V_\infty \eta}{c V_\infty}\frac{V_\infty}{P} dW_f}^{\text{Not so useful}}$$

$$ds = -\frac{V_\infty \eta}{c V_\infty}\left(\frac{W}{T}\right)\frac{dW_f}{W} = -\frac{\eta}{c}\left(\frac{T}{W}\right)^{-1}\frac{dW_f}{W}$$

Integrate

$$\int_{S_1}^{S_2} ds = -\int_{W_2}^{W_1} \frac{\eta}{c}\left(\cancel{\frac{T}{W}}\right)^{-1} \frac{dW_f}{dW} = \frac{\eta}{c}\,\frac{L}{D}\int_{W_1}^{W_2} \frac{dW_f}{W}$$

$\left(\frac{L}{D}\right)$

notice the limits and - sign

$$S_2 - S_1 = \frac{\eta}{c}\,\frac{L}{D}\left(\ln W_2 - \ln W_1\right)$$

$$\boxed{R_{prop} = \frac{\eta}{c}\left(\frac{L}{D}\right)\ln\left(\frac{W_2}{W_1}\right)}$$

Historical Breguet Equation 1910s

Assumes $\frac{L}{D}$, η, and c are constant.

If we again assume $\left(\frac{L}{D}\right)_{max} = \sqrt{\frac{1}{4C_{D_0}K}}$ & $V_\infty = \sqrt{\frac{2}{\rho}\sqrt{\frac{K}{C_{D_0}}}\frac{W}{S}}$

$$R_{prop} = \frac{\eta}{c}\sqrt{\frac{1}{4C_{D_0}K}}\ln\left(\frac{W_2}{W_1}\right) \approx \frac{\eta}{c}\sqrt{\frac{\pi ARe}{4C_{D_0}}}\ln\left(\frac{W_2}{W_1}\right)$$

To maximize the range of a prop a/c,

- $\eta \uparrow$ efficient propeller
- $c \downarrow$ low SFC fuel burn
- $ARe \uparrow$ low induced drag
- $C_{D_0} \downarrow$ low drag

$\ln\left(\frac{W_2}{W_1}\right) \uparrow$ high ratio of fuel onboard to empty weight

$$\frac{W_2}{W_1} = \frac{W_a + W_f}{W_a} \quad \text{in the max range limit}$$

Ex: Estimate the range of the Rutan Voyager.

Assume: $SFC \approx 0.4 \frac{lbf}{hp\,h}$

$\eta \approx 0.87$

$C_{D_0} \approx 340$ counts

Geometry: $b = 110.5\,ft$

$S = 363\,ft^2$

Fuel + Weight:

Gross weight $= 9700\,lbf$

Empty $= 2250\,lbf$

Aerodynamics:

$AR = 33.6$

$e \approx 0.95$ $\Rightarrow$ $k \approx 0.0996 = \frac{1}{\pi AR e}$

Range:

$$R = \frac{\eta}{c} \underbrace{\sqrt{\frac{\pi}{4} \frac{AR\,e}{C_{D_0}}}}_{\approx 27} \underbrace{\ln \frac{W_2}{W_1}}_{\ln\left(\frac{9700}{2250}\right) = 0.63}$$

$$= \frac{0.87}{} \cdot \frac{hp\,h}{0.4\,lbf} \cdot 27 \cdot 0.63 \cdot \frac{550\,lbf\,ft}{hp\,s} \cdot \frac{3600\,s}{hr} \cdot \frac{mi}{5280\,ft}$$

$= 13870$ mi

The actual Voyager flew around the Earth with no refueling. (26680 mi)

Where is our error?

1) SFC: this must actually be around $0.2 \frac{lbf}{hp\,h}$ ~ double that of a regular engine
2) $\eta \approx 90\%$ at best
3) Aero ≈ 27 L/D is quite good

$C_{D_0} = 340$ counts ?!

Aircraft Range.

$$ds = -\frac{V_\infty}{c_t}\frac{1}{T}\,dW_f$$

Mult and divide by W

$$ds = -\frac{V_\infty}{c_t}\left(\frac{W}{T}\right)\frac{dW_f}{W} \Rightarrow ds = -\frac{V_\infty}{c_t}\left(\frac{L}{D}\right)\frac{dW_f}{W}$$

Integrate

$$R = \int_{W_2}^{W_1} -\frac{V_\infty}{c_t}\left(\frac{L}{D}\right)\frac{dW_f}{W} = \int_{W_1}^{W_2}\frac{V_\infty}{c_t}\left(\frac{L}{D}\right)\frac{dW_f}{W}$$

$$\boxed{R = V_\infty\left(\frac{L}{D}\right)\frac{1}{c_t}\ln\left(\frac{W_2}{W_1}\right)}$$ Breguet Range Equation

Assumes constant V_∞, c_t, and $\left(\frac{L}{D}\right)$

Q: Given that $V_\infty \frac{L}{D}$ is specified during the flight, how does the aircraft altitude vary?

$$V_\infty\left(\frac{L}{D}\right) = V_\infty\left(\frac{C_L}{C_D}\right) = V_\infty\frac{C_L}{C_{D_0}+kC_L^2} = \sqrt{\frac{2}{\rho}\frac{1}{C_L}\left(\frac{W}{S}\right)}\cdot\frac{C_L}{C_{D_0}+kC_L^2} = \text{Constant}$$

$$= \sqrt{\frac{2}{\rho}\left(\frac{W}{S}\right)}\ \frac{\sqrt{C_L}}{\underbrace{C_{D_0}+kC_L^2}_{C_D}}$$

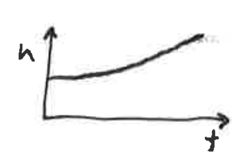
Maximized when $\frac{\sqrt{C_L}}{C_D}$ is maximized.

So for a given aircraft, $\sqrt{\frac{2}{\rho}\left(\frac{W}{S}\right)} = \text{Constant}$

$$\sqrt{\frac{2}{\rho_1}\left(\frac{W_1}{S}\right)} = \sqrt{\frac{2}{\rho_2}\left(\frac{W_2}{S}\right)} \Rightarrow \frac{\rho_2}{\rho_1} = \frac{W_2}{W_1}$$

If the aircraft weight drops by half, the density must also drop (higher altitude)

you often notice this in long-haul flights



usually stepped

Optimal $\dfrac{\sqrt{C_L}}{C_D}$

$$\frac{d}{dC_L}\left(\frac{\sqrt{C_L}}{C_{D_0}+kC_L^2}\right) = \frac{d}{dC_L}\left(C_L^{1/2}\cdot\left(C_{D_0}+kC_L^2\right)^{-1}\right) = 0$$

$$0 = \frac{1}{2}C_L^{-1/2}\cdot\left(C_{D_0}+kC_L^2\right)^{-1} + C_L^{1/2}\cdot(-1)\left(C_{D_0}+kC_L^2\right)^{-2}\left(2kC_L\right)$$

$$= \frac{\frac{1}{2}C_L^{-1/2}\left(C_{D_0}+kC_L^2\right) + C_L^{1/2}(-1)(2kC_L)}{C_{D_0}+kC_L^2} = 0$$

So,

$$\frac{1}{2}C_L^{-1/2}\left(C_{D_0}+kC_L^2\right) - C_L^{1/2}\,2kC_L = 0$$

Mult by $C_L^{1/2}$

$$\frac{1}{2}C_{D_0} + \frac{1}{2}kC_L^2 - C_L\,2k\,C_L = 0$$

Rearrange

$$\frac{1}{2}C_{D_0} = 2C_L^2k - \frac{1}{2}kC_L^2$$

$$C_{D_0} = 4C_L^2k - kC_L^2 = 3C_L^2k$$

For $\left(\dfrac{\sqrt{C_L}}{C_D}\right)_{max}$, $\boxed{C_{D_0} = 3C_L^2K}$ or $C_L = \sqrt{\dfrac{C_{D_0}}{3K}}$

Profile drag is 3 times larger than induced drag

Substitute to find

$$V_{\left(\frac{\sqrt{C_L}}{C_D}\right)_{max}} = 1.32\,V_{\left(\frac{L}{D}\right)_{max}}$$

$$\left(\frac{\sqrt{C_L}}{C_D}\right)_{max} = \frac{\sqrt{C_L}}{C_{D_0} + kC_L^2} = \frac{\sqrt[4]{\frac{C_{D_0}}{3k}}}{C_{D_0} + k\frac{C_{D_0}}{3k}} = \frac{1}{\sqrt[4]{C_{D_0}^3}\left(1 + \frac{1}{3}\right)} \cdot \frac{1}{\sqrt[4]{3k}}$$

$$= \frac{3}{4}\sqrt[4]{\frac{1}{C_{D_0}^3 3k}}$$

Returning to Range

$$R = \underbrace{V_\infty \left(\frac{L}{D}\right)}_{\substack{\text{maximized} \\ \text{when} \\ \left(\sqrt{C_L}/C_D\right)_{max}}} \frac{1}{c_t} \ln\left(\frac{W_2}{W_1}\right)$$

$$= \underbrace{\sqrt{\frac{2}{\rho}\left(\frac{W}{S}\right)}}_{V} \underbrace{\frac{3}{4}\sqrt[4]{\frac{1}{C_{D_0}^3 3k}}}_{L/D} \frac{1}{c_t} \ln\left(\frac{W_2}{W_1}\right) = \sqrt{\frac{2}{\rho}\frac{W}{S}} \frac{3}{4}\sqrt[4]{\frac{\pi ARe}{3 C_{D_0}^3}} \frac{1}{c_t} \ln\left(\frac{W_2}{W_1}\right)$$

$$V = V_{\left(\frac{\sqrt{C_L}}{C_D}\right)_{max}} = 1.32\, V_{\left(\frac{L}{D}\right)_{max}}$$

To maximize range.

- high $\frac{W}{S}$ high wing loading
- Low ρ high altitude
- low drag C_{D_0} is more important than k ($C_{D_0}^3$ vs k)
- High AR
- low fuel burn c_t
- large weight fraction of fuel

Endurance (How long can this go on .....)

$$\underbrace{\frac{dW_f}{dt}}_{\substack{\text{Change}\\ \text{in fuel}\\ \text{weight}}} = -\underbrace{C_t}_{\substack{\text{thrust specific}\\ \text{fuel consumpt}}} \cdot \underbrace{T}_{\text{thrust}} \quad \Rightarrow \quad \underbrace{dt = -\frac{dW_f}{C_t T} = -\frac{dW_f}{C_t D}}_{\text{Assume SLF}}$$

Mult and divide by L and W (same value, SLF)

$$dt = -\frac{L}{D}\frac{1}{C_t}\frac{dW_f}{W}$$

Integrate

$$\int_{t_0}^{t_f} dt = -\int_{W_{initial}}^{W_{final}} \frac{L}{D}\frac{1}{C_t}\frac{dW_f}{W} = \boxed{\int_{W_{final}}^{W_{initial}} \left(\frac{L}{D}\right)\left(\frac{1}{C_t}\right)\frac{dW_f}{W} = E}$$

For a constant $\left(\frac{L}{D}\right)$ and c_t

$$E = \left(\frac{L}{D}\right)\frac{1}{C_t}\int_{W_{final}}^{W_{initial}} \frac{dW_f}{W} = \boxed{\left(\frac{L}{D}\right)\frac{1}{C_t}\ln\left(\frac{W_{init}}{W_{final}}\right) = E}$$

Maximize:

- $\uparrow \frac{L}{D}$ fly at $\left(\frac{L}{D}\right)_{max}$
- $\downarrow c_t$ low fuel burn
- $\uparrow \frac{W_{init}}{W_{final}}$ lots of fuel

Propeller Aircraft

The propeller aircraft is slightly more complicated since C_t is a function of velocity $C_t = \frac{c V_\infty}{\eta}$. We already see that a slow speed is ideal.

$$E = \int_{W_{final}}^{W_{initial}} \left(\frac{L}{D}\right) \frac{1}{C_t} \frac{dW_f}{W} = \int_{W_{final}}^{W_{initial}} \left(\frac{L}{D}\right) \frac{\eta}{c V_\infty} \frac{dW_f}{W}$$

$$= \int_{W_{fin}}^{W_{init}} \cancel{\left(\frac{L}{D}\right)}^{C_L/C_D} \frac{\eta}{c} \underbrace{\sqrt{\frac{\rho C_L S}{2W}}}_{V^{-1}} \frac{dW_f}{W} = \int_{W_{final}}^{W_{init}} \left(\frac{C_L^{3/2}}{C_D}\right) \frac{\eta}{c} \sqrt{\frac{\rho}{2}\left(\frac{W}{S}\right)^{-1}} \frac{dW_f}{W}$$

$$= \int_{W_{final}}^{W_{init}} \left(\frac{C_L^{3/2}}{C_D}\right) \frac{\eta}{c} \sqrt{\frac{\rho S}{2}} \frac{dW_f}{W^{3/2}}$$

For a constant $\frac{C_L^{3/2}}{C_D}$, η, c, ρ_∞

$$E = \frac{\eta}{c} \left(\frac{C_L^{3/2}}{C_D}\right) \sqrt{\frac{\rho S}{2}}\, 2 \left(W_{final}^{-1/2} - W_{initial}^{-1/2}\right)$$

$$\boxed{E = \frac{\eta}{c} \left(\frac{C_L^{3/2}}{C_D}\right) \sqrt{2 \rho S} \left(W_{final}^{-1/2} - W_{initial}^{-1/2}\right)}$$

- fly at sea level (ρ highest)
- fly at $\left(\frac{C_L^{3/2}}{C_D}\right)_{max}$ which is <u>minimum power</u> speed