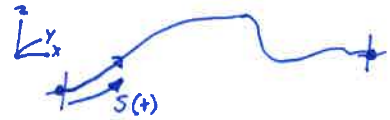
Lesson 21

Aircraft Equations of Motion

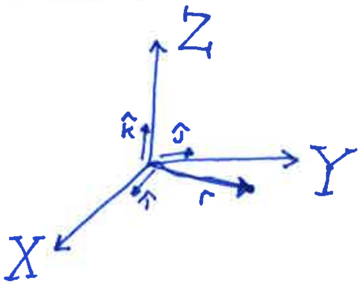
Dynamics – Particle Kinematics

The study of particle trajectories



Inertial Frame

(non-accelerating, non-rotating)



$$\vec{r} = (x,y,z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

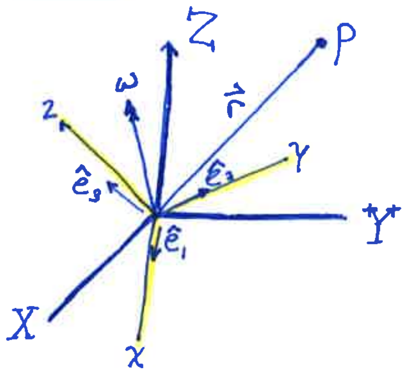
$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right) = \frac{d}{dt}(x\hat{\imath}) + \frac{d}{dt}(y\hat{\jmath}) + \frac{d}{dt}(z\hat{k})$$

$$= \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}$$

$$\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}$$

Rotating Frame

(rotating about XYZ)

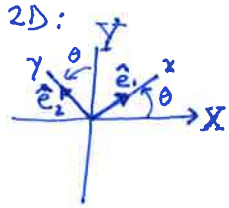


$$\vec{r} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3$$

$$\vec{V} = \frac{d\vec{r}}{dt} = \dot{x}\hat{e}_1 + \dot{y}\hat{e}_2 + \dot{z}\hat{e}_3 + x\dot{\hat{e}}_1 + y\dot{\hat{e}}_2 + z\dot{\hat{e}}_3$$

What is $\dot{\hat{e}}$? $\frac{d\hat{e}}{dt}$

In 2D:



$\hat{e}$ is a true vector

$$\hat{e}_1 = [\cos(\theta(t)),\ \sin(\theta(t))]$$

$$\hat{e}_2 = [-\sin(\theta(t)),\ \cos(\theta(t))]$$

So what is $\dot{\hat{e}}$ in 2D?

$$\dot{\hat{e}}_1 = \dot{\theta}\left[-\sin\theta(t),\ \cos\theta(t)\right] = \dot{\theta}\hat{e}_2$$

$$\dot{\hat{e}}_2 = \dot{\theta}\left[-\cos\theta(t),\ -\sin\theta(t)\right] = -\dot{\theta}\hat{e}_1$$

In general, $\dot{\hat{e}} = \vec{\omega} \times \hat{e}$

Continuing from

$$V = \underbrace{\vec{V}_r}_{r \text{ change wrt } x,y,z} + \underbrace{x(\vec{\omega}\times\hat{e}_1) + y(\omega\times\hat{e}_2) + z(\omega\times\hat{e}_3)}_{\text{how is } x,y,z \text{ changing with respect to } XYZ}$$

$$V = \vec{V}_r + \vec{\omega}\times\vec{r}$$

General Theorem

$$\dot{(\ )}_a = \dot{(\ )}_b + \vec{\omega}_{ab} \times \overrightarrow{(\ )}$$

rotation rate of b wrt a

Example: $\vec{a} = \vec{a}_r + \vec{\omega}\times\vec{V}$

Why is $x(\vec{\omega}\times\hat{e}_1) + y(\vec{\omega}\times\hat{e}_2) + z(\vec{\omega}\times\hat{e}_3) = \vec{\omega}\times\vec{r}$?

$$x\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ \hat{e}_{1x} & \hat{e}_{1y} & \hat{e}_{1z} \end{vmatrix} + y\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ \hat{e}_{2x} & \hat{e}_{2y} & \hat{e}_{2z} \end{vmatrix} + z\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ \hat{e}_{3x} & \hat{e}_{3y} & \hat{e}_{3z} \end{vmatrix}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ x\hat{e}_{1x} & x\hat{e}_{1y} & x\hat{e}_{1z} \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ y\hat{e}_{2x} & y\hat{e}_{2y} & y\hat{e}_{2z} \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ z\hat{e}_{3x} & z\hat{e}_{3y} & z\hat{e}_{3z} \end{vmatrix}$$

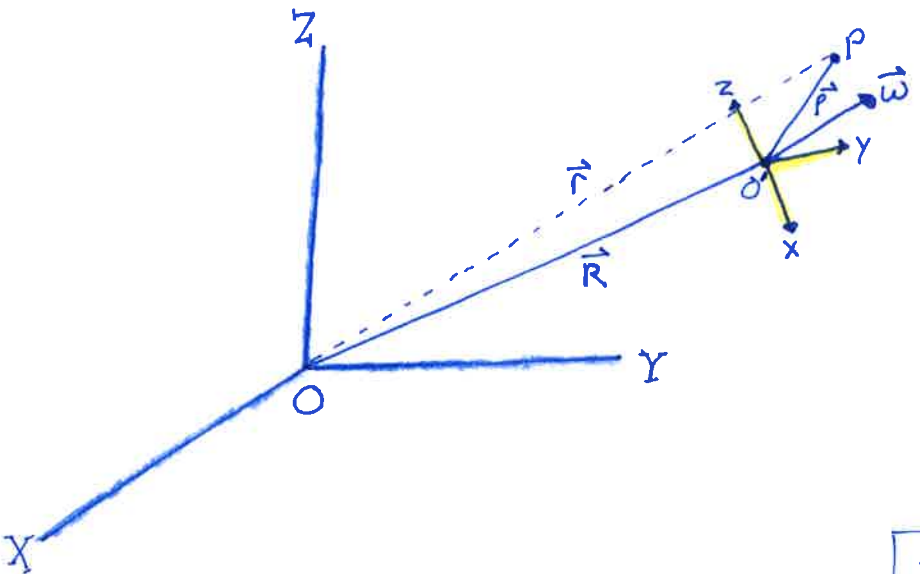
The 1st and 2nd rows are identical. We can add the 3rd row.

Plus, $x\hat{e}_{1x} + x\hat{e}_{1y} + x\hat{e}_{1z}$ is just $x\hat{e}_1$

$$x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3 = \vec{r}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ r_x & r_y & r_z \end{vmatrix} = \boxed{\vec{\omega}\times\vec{r}}$$

Rotating and Translating Reference Frame



Describe point "P"

$$\underbrace{\vec{r} = \vec{R}}_{\text{relative to } O} + \underbrace{\vec{\rho}}_{\text{relative to } O'}$$

$$V = \frac{d\vec{r}}{dt} = \dot{\vec{R}} + \dot{\vec{\rho}}$$

and $$\underbrace{\dot{\vec{\rho}}}_{\text{relative to } O} = \underbrace{\dot{\vec{\rho}}_r + \vec{\omega} \times \vec{\rho}}_{\text{relative to } O'}$$

$$\boxed{\vec{V} = \dot{\vec{R}} + \dot{\vec{\rho}}_r + \vec{\omega} \times \vec{\rho}}$$

$\dot{\vec{R}}$ = absolute velocity of O'

$\dot{\vec{\rho}}_r$ = velocity of ρ as seen from O'

Acceleration

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}\left(\dot{\vec{R}} + \dot{\vec{\rho}}_r + \vec{\omega} \times \vec{\rho}\right) \neq \cancel{\ddot{\vec{R}} + \ddot{\vec{\rho}}_r}$$

$$= \frac{d}{dt}\dot{\vec{R}} + \frac{d}{dt}\dot{\vec{\rho}}_r + \frac{d}{dt}\left(\vec{\omega} \times \vec{\rho}\right)$$

$$= \ddot{R} + \ddot{\rho}_r + \omega \times \dot{\rho} + \dot{\omega} \times \rho + \omega \times \dot{\rho} + \omega \times \omega \times \rho$$

$$\boxed{\vec{a} = \ddot{\vec{R}} + \ddot{\vec{\rho}}_r + 2\vec{\omega} \times \dot{\vec{\rho}} + \dot{\vec{\omega}} \times \vec{\rho} + \vec{\omega} \times (\vec{\omega} \times \vec{\rho})}$$

↑ O' frame acceleration ↑ local acceleration ↑ Coriolis Acceleration ↑ tangential Acceleration ↑ Centripetal Acceleration

$\frac{d\dot{\vec{R}}}{dt} = \ddot{\vec{R}}$

$\frac{d}{dt}\dot{\vec{\rho}}_r$ is <u>not</u> $\ddot{\vec{\rho}}_r$ since we want the rate of change in the O frame

$(\dot{\ }) = (\dot{\ })_r + \omega \times (\ )$

$\frac{d}{dt}(\dot{\vec{\rho}}_r) = \ddot{\rho}_r + \omega \times \dot{\rho}$

$$\frac{d}{dt}(\vec{\omega} \times \vec{\rho}) = \left(\frac{d}{dt}\vec{\omega}\right) \times \rho + \vec{\omega} \times \frac{d\rho}{dt}$$

$$= \dot{\omega} \times \rho + \vec{\omega} \times \left(\dot{\rho}_r + \omega \times \rho\right)$$

$$= \dot{\omega} \times \rho + \vec{\omega} \times \dot{\rho}_r + \omega \times (\omega \times \rho)$$

Newtonian Mechanics

Isaac Newton 1687 Mathematical Principles of Natural Philosophy (translated from Latin)

Law I: Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed.

"Inertia"

Law II: The change of momentum of a body is proportional to the impulse impressed on the body, and happens along the straight line on which that impulse is impressed

$$\boxed{F = \frac{d}{dt}(m\vec{v})}$$

Law III: To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are ~~equal~~ always equal, and directed to contrary parts.

Equal and opposite

Assumptions:

- Inertial Frame
- Velocities << speed of light
- Macro scale (not sub atomic)
- No entropy (thermodynamics)
- EMF fields

Particle Dynamics

Newton's 2nd Law + Rotating and Translating Frame

$$\sum F = m\vec{a} = m\left(\ddot{R} + \ddot{\rho}_r + 2\omega \times \dot{\rho} + \dot{\omega}\times\rho + \omega\times(\omega\times\rho)\right)$$

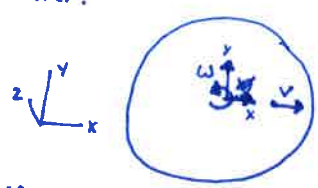
The o' frame is known: $\ddot{R}, \omega, \dot{\omega}, \rho, \dot{\rho} \Rightarrow$ find $\ddot{\rho}$

Integrate $\ddot{\rho}$ to find $\dot{\rho}$ and ρ

Often rearranged to give a non-inertial formulation

$$m\ddot{\rho}_r = \sum F - \underbrace{m\left(\ddot{R} + 2\omega\times\dot{\rho}_r + \dot{\omega}\times\rho_r + \omega\times(\omega\times\rho_r)\right)}_{\text{consider these as forces that appear when in a rotating and translation frame}}$$

Ex: On a spinning merry-go-round, you throw a ball from away from the center.



$\omega = (0, 0, \Omega)$ $\qquad \dot{\omega} = \left(0, 0, -\frac{\Omega}{100}\right)$

$\sum F \approx 0$

$\ddot{R} = 0$ (fixed mgr)

$\dot{\rho}_r = V$

$\rho \neq 0$ (away from center) $= L$

$\Rightarrow$

$$\omega\times\dot{\rho}_r = \begin{vmatrix} i & j & k \\ 0 & 0 & \Omega \\ V & 0 & 0 \end{vmatrix} = -(-V\Omega)\hat{j} = V\Omega\hat{j}$$

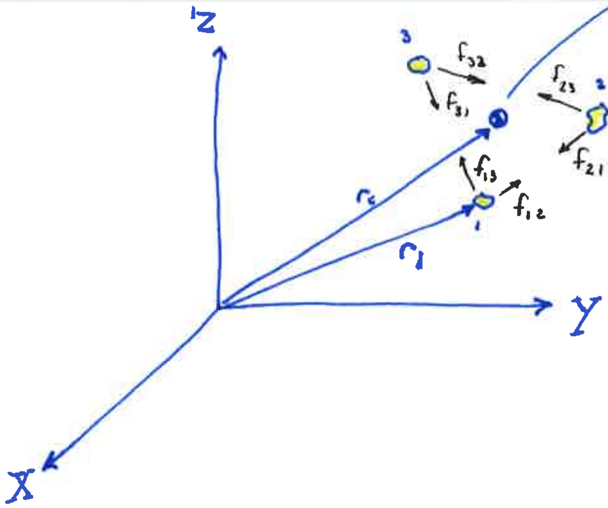
$$\dot{\omega}\times\rho = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 0 & 0 & -\frac{\Omega}{100} \\ L & 0 & 0 \end{vmatrix} = \frac{\Omega L}{100}\hat{j}$$

$$\omega\times(\omega\times\rho_r) = \omega\times(L\Omega\hat{j}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \Omega \\ 0 & L\Omega & 0 \end{vmatrix} = -\Omega^2 L\hat{i}$$

$$\ddot{\rho}_r = 0 \underbrace{- 2V\Omega\hat{j} - \frac{\Omega L}{100}\hat{j}}_{\text{from the spinning frame, the ball accelerates right}} + \Omega^2 L\hat{i}$$

System of Particles

center of mass of particles



$\vec{F_i}$ ≡ external forces

$\vec{f_{ij}}$ ≡ internal force between particles

From Newton's 3rd law:

$$\vec{f_{ij}} = -\vec{f_{ji}}$$

and

$$\vec{f_{ii}} = 0$$

Newton's 2nd Law

$$\sum F = \frac{d}{dt}(m\vec{V}) \quad \text{if } \frac{dm}{dt} = 0 \Rightarrow m\ddot{\vec{r}} = \sum F$$

for particle i of n

$$\vec{F_i} + \sum_{j=1}^{n} \vec{f_{ij}} = m_i \ddot{\vec{r}}_i$$

Now sum over $\underline{\underline{\text{all}}}$ n particles

$$\underbrace{\underbrace{\sum_{i=1}^{n} \vec{F_i}}_{\substack{\text{summation of all} \\ \text{external forces} \\ = F}} + \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n} \vec{f_{ij}}}_{\substack{=0 \text{ all internal forces cancel out.} \\ f_{34} = -f_{43} \\ \text{so} \\ f_{34} + f_{43} = 0}} = \underbrace{\sum_{i=1}^{n} m_i \ddot{\vec{r}}_i}_{\substack{\text{total mass} \equiv m = \sum m_i \\ \text{so this term is} \\ = m\ddot{r}_c}}}$$

$$F = m\ddot{\vec{r}}_c$$

We can treat a collection of masses as a single mass/system

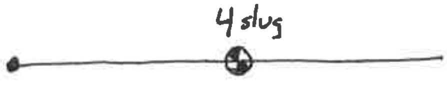
Why is $\sum m_i \ddot{\vec{r}}_i = m\ddot{\vec{r}}_c$?

1D simplified case

Where is the center of mass? CG?

$$X_{cg} = \frac{\sum M_{x=0}}{\sum m} = \frac{\text{Moment}}{\text{Force}}$$

$$= \frac{2\,\text{slug} \cdot 1\,\text{ft} + 2\,\text{slug} \cdot 5\,\text{ft}}{2\,\text{slug} + 2\,\text{slug}}$$



$$= \frac{12\,\text{slug ft}}{4\,\text{slug}} = 3\,\text{ft}$$

Now compute $\sum m_i \ddot{\vec{r}}_i$

$$\sum m_i \ddot{\vec{r}}_i = \sum m_i \frac{d^2}{dt^2}(\vec{r}) = \frac{d^2}{dt^2}\left(\sum m_i r_i\right)$$

$$= 2\,\text{slug} \cdot 1\,\text{ft} + 2\,\text{slug} \cdot 5\,\text{ft} = 12\,\text{slug ft}$$

$$= 4\,\text{slug} \cdot 3\,\text{ft}$$

Same operation as above.

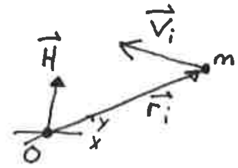
Find center of mass

$$r_c = \frac{\sum m_i r_i}{m} = \frac{\sum m_i r_i}{\sum m_i}$$

Angular Momentum

Given a particle, the angular momentum is (about a point O)

$$\vec{H} = \vec{r}_i \times m_i \dot{\vec{r}}_i$$



Ex: $\vec{V}_i = (-5, 1)\ [\frac{ft}{s}]$

$m_i = 1\,[slug]$

$\vec{r}_i = (0, 4)\ [ft]$

$$\vec{H} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 4 & 0 \\ -5 & 1 & 0 \end{vmatrix} = -\left(-20\ slug\ \frac{ft}{s}\,ft\right)\hat{k}$$

$$= 20\ \frac{slug\,ft^2}{s}\ \hat{k}$$

Time derivative

$$\dot{\vec{H}} = \frac{d}{dt}(\vec{H}) = \frac{d}{dt}(\vec{r}_i \times m_i \dot{\vec{r}}_i)$$

$$= (\vec{r}_i \times m_i \ddot{\vec{r}}_i) + \underbrace{(\dot{\vec{r}}_i \times m_i \dot{\vec{r}}_i)}_{0!!\ \text{Why?}}$$

$$\vec{A} \times \vec{A} = 0 \Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ A_x & A_y & A_z \end{vmatrix} = \overset{0}{\cancel{(A_y A_z - A_z A_y)}}\,\hat{i} + \dots$$

$$\dot{\vec{H}} = \vec{r}_i \times m_i \ddot{\vec{r}}_i$$

From the particle slide, $m_i \ddot{\vec{r}}_i = \sum \vec{F}_i + \sum_{j=1}^{n} \vec{f}_{ij}$

Subst'

$$\dot{\vec{H}}_i = \vec{r}_i \times \left(\vec{F}_i + \sum_{j=1}^{n} f_{ij}\right)$$

Sum over all particles

$$\sum \dot{\vec{H}}_i = \dot{\vec{H}} = \sum_{i=1}^{n}\left(\vec{r}_i \times \left(\vec{F}_i + \sum_{j=1}^{n} \vec{f}_{ij}\right)\right)$$

$$= \underbrace{\left(\sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i\right)}_{\text{Moment!}} + \underbrace{\left(\sum_{i=1}^{n} \vec{r}_i \times \sum_{j=1}^{n} \vec{f}_{ij}\right)}_{\underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n} \vec{r}_i \times \vec{f}_{ij}}_{0}}$$

$$\boxed{\dot{\vec{H}} = M}$$