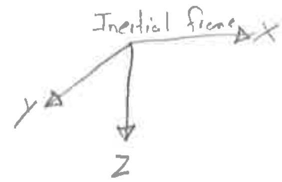
Lesson 21 part 2

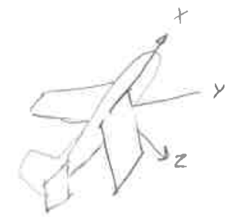
Rigid Body Eqs' of motion

Rigid Body Equations of Motion

Single, non-flexing object

moving and rotating in space.





In the inertial frame, apply Newton's law

$$\underbrace{\frac{d}{dt}(m\vec{V}) = \sum \vec{F}}_{\text{Translation, Force}} \qquad \underbrace{\frac{d}{dt}(\vec{H}) = \sum \vec{M}}_{\text{Rotation, Moments}}$$

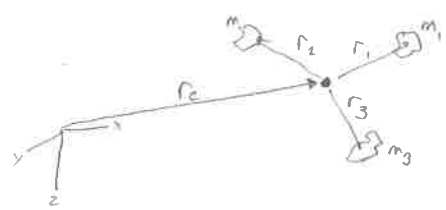
When $\frac{d}{dt}m = 0$

$$\underbrace{\begin{matrix} m\ddot{x} = F_x & \dot{H}_x = L \\ m\ddot{y} = F_y & \dot{H}_y = M \\ m\ddot{z} = F_z & \dot{H}_z = N \end{matrix}}_{\text{6 degrees of freedom "6DOF"}}$$

6 degrees of freedom
"6DOF"

(3 translation + 3 rotation)

Given a collection of particles creating an object,



The vector r_c is to the mass center, which we call the Center of Gravity

r_c is the location where

$$\sum m_i r_i = 0$$

The force/translation is simple

$$m\ddot{r}_c = \sum F$$

Expand the rotation/moment equation

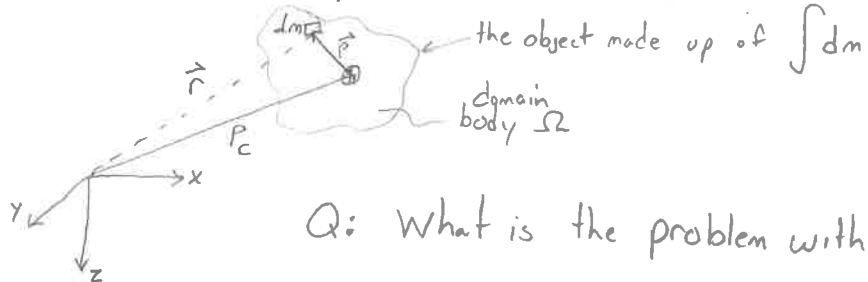
$$\Sigma M = \frac{d}{dt} H = \frac{d}{dt}\left(r_i \times m_i \dot{r}_i\right)$$

$$= \underbrace{\left(\dot{r}_i \times m_i \dot{r}_i\right)}_{=0} + \left(r_i \times m_i \ddot{r}_i\right)$$

$$= r_i \times m_i \ddot{r}_i \qquad = r_i \times \ddot{r}_i\, m_i$$

For one small mass dm, the small moment would be

$$d\dot{H} = r_i \times \ddot{r}_i\, dm$$



Q: What is the problem with this formulation?

A: As the body rotates, the position ρ of each dm continuously changes (even though the body is non-flexing and in the body frame nothing is moving).

Finding $\dot{H} = \int_\Omega d\dot{H} = \int_\Omega r \times \ddot{r}\, dm$ is

time dependent.

This integral varies with vehicle orientation ϕ, θ, ψ

Q: What is the solution?

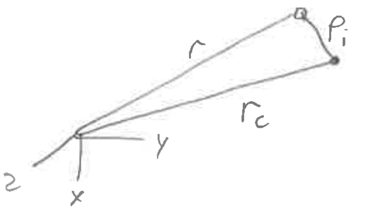
A: Pick a reference frame where the mass doesn't move with respect to the frame.

The body frame

Q: But this isn't an inertial frame.

A: Correct, extra terms will appear. But, at least the integrals won't vary.

Take a reference point about the center of Mass.



$r = r_c + \rho_i$

Angular Momentum i.e.

$H = r \times \dot{r} m$ or in terms of small masses $H = r_i \times \dot{r}_i m_i$

Substitute

$$H = (r_c + \rho_i) \times \dot{r}_i m_i = (r_c + \rho_i) \times (\dot{r}_c + \dot{\rho}_i) m_i$$

$$= \underbrace{r_c \times \dot{r}_c m_i}_{\Sigma = r_c \times \dot{r}_c m} + \underbrace{r_c \times \dot{\rho}_i m_i}_{\Sigma \rho_i m_i = 0} + \underbrace{\underbrace{\rho_i \times \dot{r}_c m_i}_{= -\dot{r}_c \times \rho_i m_i}}_{\Sigma \dot{r}_c \times \rho_i m_i = 0} + \underbrace{\rho_i \times \dot{\rho}_i m_i}_{\Sigma = H_c}$$

$$= r_c \times \dot{r}_c m + H_c$$

Take $\frac{d}{dt}$ of H

1 $$\dot{H} = \frac{d}{dt}(r_c \times \dot{r}_c m) + \dot{H}_c = \underbrace{\dot{r}_c \times \dot{r}_c m}_{0} + r_c \times \underbrace{\ddot{r}_c m}_{F} + \dot{H}_c$$

$$= r_c \times F + \dot{H}_c$$

and

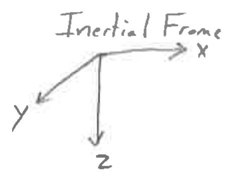
2 $$\dot{H} = M = r_j \times F_j = (r_c + \rho_j) \times F_j = r_c \times F_j + \rho_i \times F_j$$

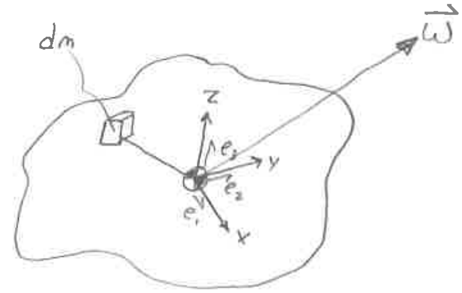
Equate the equations 1 and 2

$$\overbrace{r_c \times F + \dot{H}_c = \dot{H}}^{1} \overbrace{= r_c \times F + \rho_j \times F_j}^{2} \Rightarrow \boxed{\dot{H}_c = \rho_i \times F_i}$$

We can center the angular momentum reference point to the CG!

Body Frame Reference





The angular momentum is $dH \equiv r \times dm\, \dot{r} = r \times \dot{r}\, dm$

where r and $\dot{r}$ are in the inertial frame

or

$$\underline{dH_c \doteq \rho \times \dot{\rho}\, dm}$$ where ρ and $\dot{\rho}$ are still in the inertial frame, but measuring a distance on the body.

So common that we will rename this H below this point

$$H = \int \rho \times \dot{\rho}\, dm$$

From our previous general rotation frame result

$$(\dot{\rho})_{\text{Inertial}} = \underbrace{(\dot{\rho})_{\text{Body}}}_{\substack{\text{No motion in body frame} \\ =0}} + \omega \times r_{\text{Body}}$$

$$= \int \rho \times (\omega \times \rho)\, dm$$

$$\vec{\rho} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3 \quad \text{and} \quad \vec{\omega} = p\hat{e}_1 + q\hat{e}_2 + r\hat{e}_3$$

We could do this $\rho \times (\omega \times \rho)$, However $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\underbrace{(\vec{A} \cdot \vec{C})}_{\substack{\text{dot product} \\ = \text{Scalar}}} - C\underbrace{(\vec{A} \cdot \vec{B})}_{\text{dot product}}$

$$\rho \times (\omega \times \rho) = \omega(\rho \cdot \rho) - \rho(\rho \cdot \omega)$$

and

$$\rho \cdot \rho = x^2 + y^2 + z^2$$
$$\rho \cdot \omega = xp + yq + zr$$

$$H = \int \begin{pmatrix} p(x^2+y^2+z^2)\hat{e}_1 + q(x^2+y^2+z^2)\hat{e}_2 + r(x^2+y^2+z^2)\hat{e}_3 \\ -x(xp+yq+zr)\hat{e}_1 - y(xp+yq+zr)\hat{e}_2 - z(xp+yq+zr)\hat{e}_3 \end{pmatrix} dm$$

x^2p terms cancel

$$\vec{H} = \int \left(p(y^2+z^2) - xyq - xzr\right)\hat{e}_1\, dm + \left(q(x^2+z^2) - xyp - yzr\right)\hat{e}_2\, dm + \left(r(x^2+y^2) - xzp - yzq\right)\hat{e}_3\, dm$$

In a matrix format,

$$\vec{H} = \underbrace{\begin{bmatrix} \int(y^2+z^2)\,dm & -\int xy\,dm & -\int xz\,dm \\ -\int xy\,dm & \int(x^2+z^2)\,dm & -\int yz\,dm \\ -\int xz\,dm & -\int yz\,dm & \int x^2+y^2\,dm \end{bmatrix}}_{\text{Inertia Matrix}} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

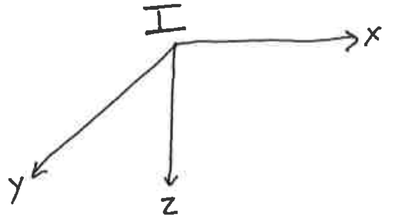
$$= \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

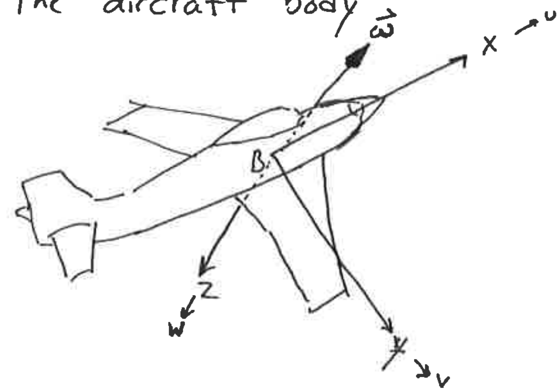
Notice that this matrix is symmetrical

The scaler forms are:

$$\boxed{\begin{aligned} H_x &= I_x p - I_{xy} q - I_{xz} r \\ H_y &= -I_{xy} p + I_y q - I_{yz} r \\ H_z &= -I_{xz} p - I_{yz} q + I_z r \end{aligned}}$$

Fix the rotating - translating frame to the aircraft body





From $\dot{(\ )}_I = \dot{(\ )}_B + \omega \times (\ )$

$$\vec{F} = m \dot{V}_c\Big|_B + m(\omega \times V_c)$$

$$\vec{M} = \frac{d\vec{H}}{dt}\Big|_B + \cancel{m}\ \omega \times \vec{H}$$

Expanding gives

$$F_x = m\dot{u} + m \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & r \\ u & v & w \end{vmatrix}_{\hat{i}} = m\dot{u} + m(qw - rv) = m(\dot{u} + qw - rv)$$

$$F_y = m(\dot{v} + ru - pw)$$

$$F_z = m(\dot{w} + pv - qu)$$

Expanding gives

$$M_x = L = \dot{H}_x + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & r \\ H_x & H_y & H_z \end{vmatrix}_{\hat{i}} = \dot{H}_x + qH_z - rH_y$$

$$M_y = M = \dot{H}_y + rH_x - pH_z$$

$$M_z = N = \dot{H}_z + pH_y - qH_x$$

But we can substitute the inertia matrix form for $\vec{H}$.

$$H_x = I_x p - I_{xy} q - I_{xz} r \Rightarrow \dot{H}_x = I_x \dot{p} - I_{xy} \dot{q} - I_{xz} \dot{r}$$

$$H_y = -I_{xy} p + I_y q - I_{yz} r \Rightarrow \dot{H}_y = -I_{xy} \dot{p} + I_y \dot{q} - I_{yz} \dot{r}$$

$$H_z = -I_{xz} p - I_{yz} q + I_z r \Rightarrow \dot{H}_z = -I_{xz} \dot{p} - I_{yz} \dot{q} + I_z \dot{r}$$

- subst into $\dot{H}_x + qH_z - rH_y = L$

$$L = \underbrace{I_x \dot{p} - I_{xy} \dot{q} - I_{xz} \dot{r}}_{\dot{H}_x} + \underbrace{-qp\, I_{xz} - q^2 I_{yz} + qr\, I_z}_{qH_z}$$

$$\underbrace{+ rp\, I_{xy} - rq\, I_y + r^2 I_{yz}}_{-rH_y}$$

simplify

$$L = I_x \dot{p} - I_{xy} \dot{q} - I_{xz} \dot{r} - qp\, I_{xz} + rp\, I_{xy} + (r^2 - q^2) I_{yz} + qr\, I_z - rq\, I_y$$

$$M = -I_{xy} \dot{p} + I_y \dot{q} - I_{yz} \dot{r} + rp\, I_x - rq\, I_{xy} - r^2 I_{xz} + p^2 I_{xz} + pq\, I_{yz} - pr\, I_z$$

$$N = -I_{xz} \dot{p} - I_{yz} \dot{q} + I_z \dot{r} - p^2 I_{xy} + pq\, I_y - pr\, I_{yz} - qp\, I_x + q^2 I_{xy} + qr\, I_{xz}$$

We can solve for the body rates.

$$I_x \dot{p} - I_{xy}\dot{q} - I_{xz}\dot{r} = L + (\cdots)$$
$$-I_{xy}\dot{p} + I_y \dot{q} - I_{yz}\dot{r} = M + (\cdots)$$
$$-I_{xz}\dot{p} - I_{yz}\dot{q} + I_z \dot{r} = N + (\cdots)$$

This is written in a matrix format as

$$\underbrace{\begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{bmatrix}}_{\substack{\text{Inertia} \\ \text{Matrix}} = I_n} \begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{bmatrix} \begin{matrix} L \\ M \\ N \end{matrix} + (\cdots) \end{bmatrix}$$

Thus,

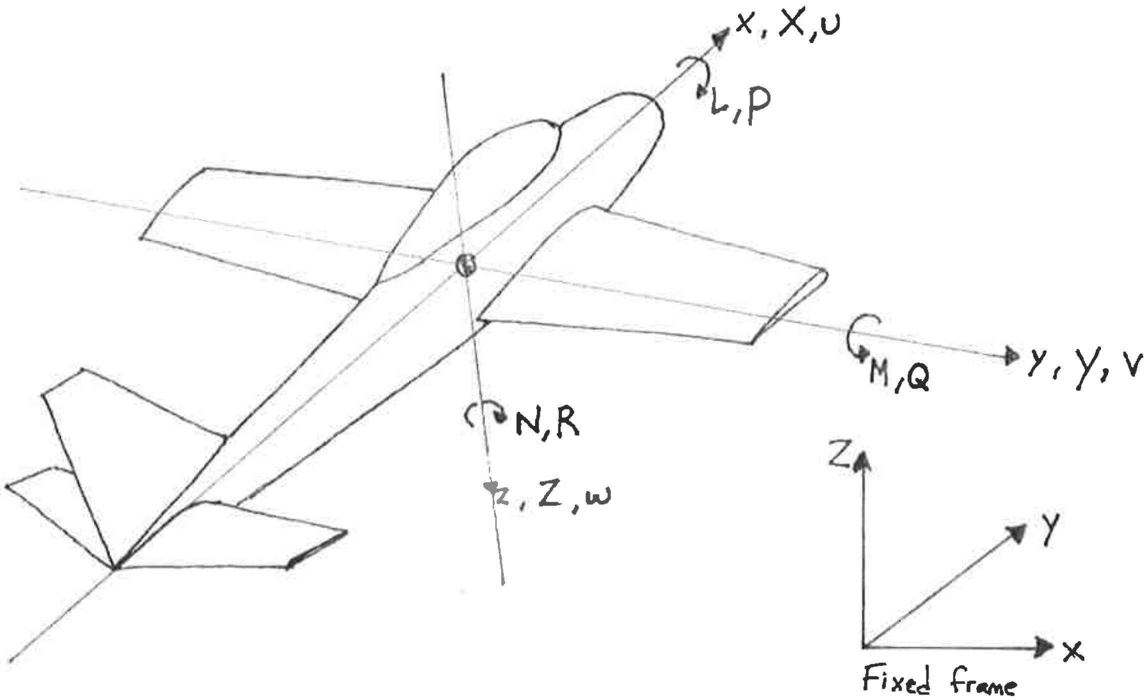
$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = [I_n]^{-1} \left[\begin{pmatrix} L \\ M \\ N \end{pmatrix} + (\cdots) \right]$$

And

$$m \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} F_x - qw + rv \\ F_y - ru - pw \\ F_z - pv - qu \end{pmatrix}$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = m^{-1} \left(\cdots \right)$$

Aircraft Coordinate System



x, y, z	Aircraft "stability" frame location
X, Y, Z	Forces in stability frame
u, v, w	Velocity in stability frame
L, M, N	moment in stability frame
P, Q, R	Angulor velocities (roll, pitch, yaw)
ϕ, θ, ψ	Euler angles (orientation)

Warning:

Outside of aerodynamics, the x axis is usually down the length of the a/c.

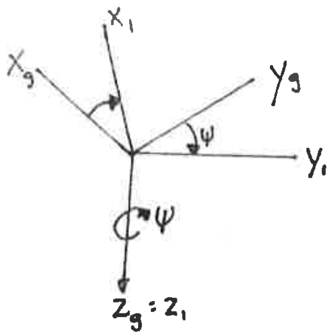
$$X_{loft} = -X_{aero\ stability}$$
$$Y_{loft} = Y_{aero\ stability}$$
$$Z_{loft} = -Z_{aero\ stability}$$

Euler Angles ϕ, θ, ψ (order dependent!)

global to local ~~local to global~~: yaw (ψ), pitch (θ), roll (ϕ)

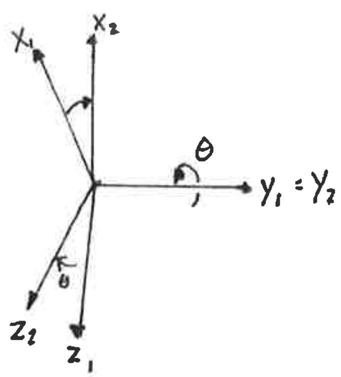
local to global ~~global to local~~: roll (ϕ), pitch (θ), yaw (ψ)

Yaw



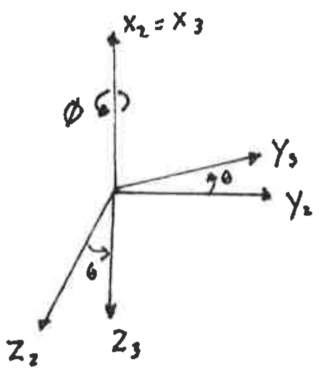
$$\left(\bar{X}_1\right) = \underbrace{\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{T_\psi^{lg}} \left(\bar{X}_G\right)$$

pitch



$$\bar{X}_2 = \underbrace{\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}}_{T_\theta^{lg}} \bar{X}_1$$

Roll



$$\bar{X}_3 = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}}_{T_\phi^{lg}} \bar{X}_2$$

Global

$$\bar{X}_G = \underbrace{T_\psi^{lg}\, T_\theta^{lg}\, T_\phi^{lg}}_{T^{lg}}\ \bar{X}_L$$

$$T^{lg} = \begin{bmatrix} C_\theta C_\psi & S_\phi S_\theta C_\psi - C_\phi S_\psi & C_\phi S_\theta C_\psi + S_\phi S_\psi \\ C_\theta S_\psi & S_\phi S_\theta S_\psi + C_\phi C_\psi & C_\phi S_\theta S_\psi - S_\phi C_\psi \\ -S_\theta & S_\phi C_\theta & C_\phi C_\theta \end{bmatrix}$$

A beautiful property of transformation matrices is

$$T^{gl} = \left(T^{lg}\right)^{-1} = T^{lg\,T}$$

The inverse is the transpose!

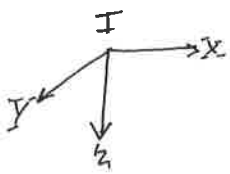
Ex:

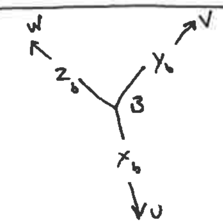
Given a global frame velocity of $\bar{V} = (1, 0, 0)$ and the orientation Euler angles of $(\phi, \theta, \psi) = (0, 10°, 10°)$, what is the body frame velocity vector (u, v, w)?

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = T^{gl} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = T^{lg\,T} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} C_\theta C_\psi \\ S_\phi S_\theta C_\psi - C_\phi S_\psi \\ C_\phi S_\theta C_\psi + S_\phi S_\psi \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0.9698 \\ -0.1413 \\ 0.1986 \end{pmatrix}$$

Body Frame to Inertial frame Velocities





$\Rightarrow$ T^{lg} takes us from the local frame (body) to the global frame

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = T^{lg} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Ex: An aircraft is pointed straight up: $\theta = 90°$. Find $\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix}$ if $U = V_\infty$.

$\cos(90) = 0$
$\sin(90) = 1$

$V = 0$
$W = 0$

$$T^{lg} = \begin{bmatrix} \cancelto{0}{C_\theta C_\psi} & \cancelto{0}{S_\phi} S_\theta C_\psi - C_\phi \cancelto{0}{S_\psi} & \cancelto{1}{C_\phi} \cancelto{1}{S_\theta} \cancelto{1}{C_\psi} + \cancelto{0}{S_\phi S_\psi} \\ C_\theta \cancelto{0}{S_\psi} & \cancelto{0}{S_\phi S_\theta S_\psi} + \cancelto{1}{C_\phi} \cancelto{1}{C_\psi} & \cancelto{0}{C_\phi S_\theta S_\psi} - \cancelto{0}{S_\phi C_\psi} \\ -\cancelto{1}{S_\theta} & S_\phi \cancelto{0}{C_\theta} & \cancelto{0}{C_\phi C_\theta} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{pmatrix} V_\infty \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -V_\infty \end{pmatrix}$$ ✓

remember