Lesson 22

Solving systems of ODEs.

rdinary
ifferential
quations

Or, learning to see the matrix....

How can a coupled set of non-linear differential equations be solved?

Ex:

$$m\ddot{x} = T - D$$

$\ddot{x}$ represents $\frac{d^2x}{dt^2} \equiv$ acceleration $= \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}V = T - D$

and

$$V = \frac{dx}{dt} = \dot{x}$$

Thus,

$$\frac{dV}{dt} = T - D \xRightarrow{\text{Mult by } dt} \text{Integrate} \Rightarrow \int_{V_a}^{V_b} dV = \int_{t_a}^{t_b} (T-D)\,dt$$

$$V_b - V_a = \int_{t_a}^{t_b} (T-D)\,dt$$

and

$$\frac{dX}{dt} = V \Rightarrow \int_{X_a}^{X_b} dX = \int_{V_a}^{V_b} v\,dt \Rightarrow X_b - X_a = \int_{V_a}^{V_b} v\,dt$$

In general, you track "states" and the time derivative of states.

Ex

$$S = \begin{pmatrix} V \\ X \end{pmatrix} \equiv \begin{pmatrix} \text{Velocity} \\ \text{distance} \end{pmatrix} \Rightarrow \dot{S} \equiv \frac{dS}{dt} = \begin{pmatrix} \frac{dV}{dt} \\ \frac{dx}{dt} \end{pmatrix}$$

So, $\dot{S}$ might be

$$\dot{S} = \begin{pmatrix} \frac{dV}{dt} \\ \frac{dx}{dt} \end{pmatrix} = \begin{pmatrix} T - D \\ v \end{pmatrix} \text{ or } \begin{pmatrix} \frac{dV}{dt} \\ \frac{dx}{dt} \end{pmatrix} = \begin{pmatrix} T - D \\ v \end{pmatrix}$$

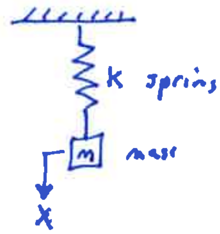
Generic:

$$\dot{S} = f(S, t) + \text{Inputs}$$

Sometimes,

$$\dot{S} = [A]\, S$$ ← the time derivative of states depends only on the states' values.

Ex:



$$m\ddot{x} + kx = 0$$

or

$$\underbrace{\ddot{x} = -\frac{k}{m}x}_{\text{Notice not in the form } \dot{s} = As}$$

But, if we track the states

$$S = \begin{pmatrix} x \\ v \end{pmatrix} \Rightarrow \dot{S} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \qquad \frac{dx}{dt} \nearrow V \qquad \frac{dv}{dt} \nearrow -\frac{k}{m}x$$

$$\underbrace{\dot{S} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}}_{\text{This is a linear function of states and is thus a linear ODE}}$$

For linear ODEs, the solution to $\dot{s} = as$ is $s(t) = S_0 e^{\lambda t}$

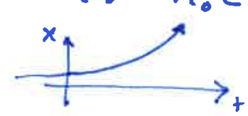
Substitute to give $\dot{s} = S_0 \lambda e^{\lambda t} = as = a S_0 e^{\lambda t}$

or

$$\lambda = a$$

Ex:

$$\dot{x} = 5x \Rightarrow \lambda = 5 \Rightarrow x(t) = X_0 e^{5t}$$



For a 2^{nd} order ODE, the roots of the characteristic equation determine λ.

Ex:

$$m\ddot{x} + c\dot{x} + kx = 0$$

substitute $x = Ae^{\lambda t}$

$$m\lambda^2 Ae^{\lambda t} + c\lambda Ae^{\lambda t} + kAe^{\lambda t} = 0$$

or

$$m\lambda^2 + c\lambda + k = 0 \Rightarrow \underline{\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0}$$

Roots are: (quadratic equ)

$$a = 1 \quad b = \frac{c}{m} \quad c = \frac{k}{m}$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{c}{2m} \pm \frac{1}{2}\sqrt{\frac{c^2}{m^2} - 4\frac{k}{m}}$$

Solution

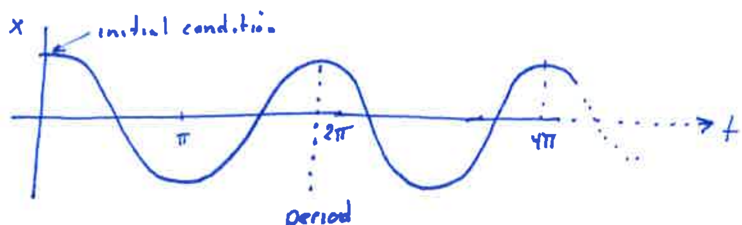
$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

For a pure $\underbrace{\text{spring}}_{k}$ + $\underbrace{\text{mass}}_{m}$, $\lambda = \pm\sqrt{-\frac{k}{m}}$

$$x(t) = Ae^{\sqrt{-\frac{k}{m}}\,t} + Be^{-\sqrt{-\frac{k}{m}}\,t} \qquad \sqrt{-1} \equiv i$$

$$= Ae^{i\sqrt{\frac{k}{m}}\,t} + Be^{-i\sqrt{\frac{k}{m}}\,t} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$= C \cdot \cos\left(\sqrt{\frac{k}{m}}\,t\right)$$



$$2\pi = \sqrt{\frac{k}{m}}\,t \Rightarrow t_{period} = 2\pi\sqrt{\frac{m}{k}} \Rightarrow f = \frac{\omega}{2\pi} = \sqrt{\frac{k}{m}}\frac{1}{2\pi}$$

For a system, the eigenvalue of the linear matrix determines λ

$$\dot{S} = AS \Rightarrow \lambda = \text{eig}(A)$$

$$P = \text{eigvectors}(\lambda_1, \lambda_2, \ldots)$$

In fact, $P^{-1}AP = \Lambda$

$\nwarrow$ diagonal matrix of λ

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix}$$

So, $S = P\underset{\nwarrow}{X}$ orthogonal vector of states and $\dot{S} = P\dot{V}$

$$\dot{S} = P\dot{X} = AS = APX$$

Now premultiply by P^{-1}

$$\underbrace{P^{-1}P}_{I}\dot{X} = \underbrace{P^{-1}AP}_{\Lambda}X \Rightarrow \dot{X} = \Lambda X \Rightarrow \begin{array}{l} \dot{X}_1 = \lambda_1 X_1 \\ \dot{X}_2 = \lambda_2 X_2 \\ \vdots \\ \dot{X}_3 = \lambda_n X_3 \end{array}$$

Ex:

$$\dot{S} = \underbrace{\begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix}}_{A} S \Rightarrow \begin{array}{l} \dot{s}_1 = 8s_2 \\ \dot{s}_2 = 2s_1 \end{array}$$

- eigenvalues

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 8 \\ 2 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 16 = 0$$

$$\lambda = \pm 4$$

$$\Lambda = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}$$

- eigenvectors

$$Av_1 = \lambda v_1 \Rightarrow \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix} v_1 = 4v_1$$

assume $v_1 = \begin{pmatrix} 1 \\ a \end{pmatrix} \Rightarrow \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ a \end{pmatrix} - 4\begin{pmatrix} 1 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$8a - 4 = 0$ top row <u>only</u>

$$a = \tfrac{1}{2} \Rightarrow v_1 = \begin{pmatrix} 1 \\ \tfrac{1}{2} \end{pmatrix}$$

$$Av_2 = \lambda v_2 \Rightarrow \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix} v_2 = -4 v_2$$

assume $v_2 = \begin{pmatrix} 1 \\ a \end{pmatrix} \Rightarrow \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ a \end{pmatrix} = -4 \begin{pmatrix} 1 \\ a \end{pmatrix} \Rightarrow 8a = -4 \Rightarrow a = -\frac{1}{2}$

$$v_2 = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \qquad P^{-1} = \frac{\text{flip diagonal} + \text{- sign off diagonal}}{\det(P)} = \frac{\begin{bmatrix} -\frac{1}{2} & -1 \\ -\frac{1}{2} & 1 \end{bmatrix}}{-\frac{1}{2} - \frac{1}{2}} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & -1 \end{bmatrix}$$

For $\dot{S} = AS \Rightarrow \dot{X} = \Lambda X = \begin{matrix} \dot{x}_1 = 4x_1 \\ \dot{x}_2 = -4x_2 \end{matrix} \Rightarrow \begin{matrix} x_1(t) = x_{10} e^{4t} \\ x_2(t) = x_{20} e^{-4t} \end{matrix}$

also,

$$X = P^{-1} S \Rightarrow X_0 = P^{-1} S_0 = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{pmatrix} S_{10} \\ S_{20} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} S_{10} + S_{20} \\ \frac{1}{2} S_{10} - S_{20} \end{pmatrix}$$

so,

$$X = \begin{pmatrix} \left(\frac{1}{2} S_{10} + S_{20}\right) e^{4t} \\ \left(\frac{1}{2} S_{10} - S_{20}\right) e^{-4t} \end{pmatrix}$$

And

$$S = PX = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} X = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{pmatrix} \left(\frac{1}{2} S_{10} + S_{20}\right) e^{4t} \\ \left(\frac{1}{2} S_{10} - S_{20}\right) e^{-4t} \end{pmatrix}$$

$$\boxed{\begin{aligned} S(t) = {} & \left(\tfrac{1}{2} S_{10} + S_{20}\right) e^{4t} + \left(\tfrac{1}{2} S_{10} - S_{20}\right) e^{-4t} \\ & \tfrac{1}{2}\left(\tfrac{1}{2} S_{10} + S_{20}\right) e^{4t} - \tfrac{1}{2}\left(\tfrac{1}{2} S_{10} - S_{20}\right) e^{-4t} \end{aligned}}$$

Numerical Solution:

$$\dot{S} = f(S, t, u, \ldots)$$

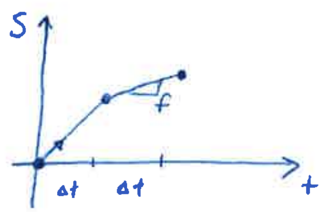
$$\frac{dS}{dt} \Rightarrow dS = f(s,t,u)\,dt$$

Integrate

$$\int_{S_0}^{S_1} dS = \int_{t_0}^{t_1} f(s,t,u)\,dt \Rightarrow S_1 - S_0 = \int_{t_0}^{t_1} f(s,t,u)\,dt$$

Simple constant f

$$S_1 - S_0 = \Delta t \cdot f(S_0)$$



Explicit Euler

System

$$S = \begin{pmatrix} x \\ v \end{pmatrix} \qquad \dot{S} = \begin{pmatrix} v \\ F/m \end{pmatrix}$$

Say $F/m = 32\ ft/s^2$

t	x	v	F/m
0	0	0	$32\ ft/s^2$
0.5			

$$x_{new} = x_{old} + \Delta t \cdot v$$

$$v_{new} = v_{old} + \Delta t \cdot F/m$$