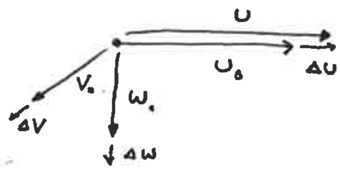
Lesson 23

Small Disturbance Theory

FSAC, 3.5

Stability Derivatives

Small deviations around a steady flight condition:



Aircraft Dynamics States

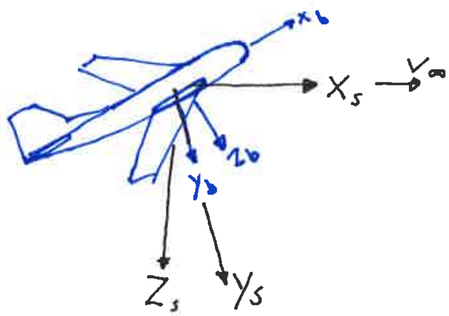
$$\left.\begin{aligned} U &= U_0 + \Delta U \\ V &= V_0 + \Delta V \\ W &= W_0 + \Delta W \end{aligned}\right\} \text{Body Frame Velocity}$$

$$\left.\begin{aligned} p &= p_0 + \Delta p \\ q &= q_0 + \Delta q \\ r &= r_0 + \Delta r \end{aligned}\right\} \text{Body frame rotation rate}$$

$$\left.\begin{aligned} \mathbf{X} &= \mathbf{X}_0 + \Delta \mathbf{X} \\ \mathbf{Y} &= \mathbf{Y}_0 + \Delta \mathbf{Y} \\ \mathbf{Z} &= \mathbf{Z}_0 + \Delta \mathbf{Z} \end{aligned}\right\} \text{Forces}$$

$$\left.\begin{aligned} L &= L_0 + \Delta L \\ M &= M_0 + \Delta M \\ N &= N_0 + \Delta N \end{aligned}\right\} \text{Moments}$$

For a typical (or reference) flight condition



Change the reference frame to X_s Y_s Z_s by aligning the x axis along V_∞

$$\left.\begin{aligned} V_0 &= 0 \\ W_0 &= 0 \end{aligned}\right\} \text{no sideslip}$$

$$\left.\begin{aligned} p_0 &= 0 \quad \text{no steady roll rate} \\ q_0 &= 0 \quad \text{no " pitch rate} \\ r_0 &= 0 \quad \text{no " yaw "} \end{aligned}\right\} \Rightarrow \begin{aligned} \phi_0 &= 0 \\ \psi_0 &= 0 \end{aligned}$$

Keep in θ!

Substitute into the equations of motion (Lesson 21 part 3)

$$\begin{pmatrix} \dot{U} \\ \dot{V} \\ \dot{W} \end{pmatrix} = \begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \begin{aligned} &\leftarrow \text{gravity } X_g = -mg\sin\theta \\ \\ &\leftarrow \text{gravity } Z_g = +mg \cos\theta \cos\phi \end{aligned}$$

look at top row for $\dot{U}$

$$\frac{d}{dt}(U_0 + \Delta U) = \dot{U} = (r_0 + \Delta r)(V_0 + \Delta V) - (q_0 + \Delta q)(W_0 + \Delta W) + \frac{1}{m}(X_0 + \Delta X - mg\sin\theta)$$

$$\underbrace{\frac{d}{dt}(U_0)}_{=0} + \frac{d}{dt}(\Delta U) = \underbrace{r_0 V_0}_{0} + \underbrace{r_0 \Delta V}_{0} + \Delta r \underbrace{V_0}_{0} + \underbrace{\Delta r \Delta V}_{HOT} - \underbrace{q_0 W_0}_{0} - \underset{0}{q_0} \Delta W - \Delta q \underset{0}{W_0} - \underbrace{\Delta q \Delta W}_{HOT} + \frac{X_0}{m} + \frac{\Delta X}{m} - g\sin(\theta + \Delta\theta)$$

Now drop the higher order terms and zero terms ($r_0 = q_0 = p_0 = 0$) etc

$a_0 + \Delta a$ assumes $a_0 \gg \Delta a$

so $(a_0 + \Delta a)(a_0 + \Delta a) = a_0^2 + 2a_0 \Delta a + \underbrace{\Delta a^2}_{\text{very small H.O.T.}}$

$\approx a_0^2 + 2a_0 \Delta a$

Simplifies to

$$\underbrace{\frac{d(\Delta U)}{dt}}_{\Delta \dot{U}} = \frac{X_0}{m} + \frac{\Delta X}{m} - g\sin(\theta_0 + \Delta\theta)$$

Also, there is an identity for trig "angle sums"

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

Apply $\theta + \Delta\theta$

$$\sin(\theta_0 + \Delta\theta) = \sin(\theta_0)\underbrace{\underbrace{\cos(\Delta\theta)}_{\Delta\theta \approx 0}}_{1} + \cos(\theta_0)\underbrace{\underbrace{\sin(\Delta\theta)}_{\Delta\theta \approx 0}}_{\Delta\theta}$$

$$= \sin\theta_0 + \Delta\theta\cos\theta_0$$

So the equation of motion is

$$\boxed{\Delta\dot{U} = \frac{X_0}{m} + \frac{\Delta X}{m} - g\sin\theta_0 - g\Delta\theta\cos\theta_0}$$

This is the linearized form of the flight dynamics.

Apply Small Disturbance to the Pitch Moment Equations

$$M = I_y \dot{q} + rp(I_x - I_y) + I_{xz}(p^2 - r^2)$$

$M = M_0 + \Delta M$

$q = q_0 + \Delta q$

$r = r_0 + \Delta r$

$p = p_0 + \Delta p$

Substitute into above

$$M_0 + \Delta M = I_y \frac{d}{dt}(q_0 + \Delta q) + (r_0 + \Delta r)(p_0 + \Delta p)(I_x - I_y) + I_{xz}\left((p_0 + \Delta p)^2 - (r_0 + \Delta r)^2\right)$$

$$= I_y \Delta\dot{q} + (I_x - I_y)\left[r_0 p_0 + r_0 \Delta p + p_0 \Delta r + \Delta r \Delta p\right]$$

$$+ I_{xz}\left[p_0^2 + 2p_0 \Delta p + (\Delta p)^2 - r_0^2 - 2r_0 \Delta r - (\Delta r)^2\right]$$

Assuming unaccelerated non steady rotating flight

$r_0 = p_0 = 0$

$$\underbrace{M_0 + \Delta M}_{} = I_y \Delta\dot{q} + (I_x - I_y)\left[0 + 0 + 0 + 0\right] + I_{xz}\left[0 + 0 + 0 + 0 + 0 + 0\right]$$

$$\boxed{I_y \Delta\dot{q} = M_0 + \Delta M}$$

This is the linearized form that we are familiar with.

For steady flight, $M_0 = 0$

$$\boxed{I_y \Delta\dot{q} = \Delta M}$$

Euler Rates

$$\dot{\theta} = q\cos\phi - r\sin\phi$$

$$\begin{aligned} \theta &= \theta_0 + \Delta\theta \\ q &= q_0 + \Delta q \\ \phi &= \phi_0 + \Delta\phi \\ r &= r_0 + \Delta r \end{aligned}$$

Linearize

$$\frac{d}{dt}(\theta_0 + \Delta\theta) = (q_0 + \Delta q)\underbrace{\cos(\phi_0 + \Delta\phi)}_{\cos\phi_0 - \Delta\phi\sin\phi_0} - \cancelto{(r_0+\Delta r)}{r}\underbrace{\sin(\phi_0 + \Delta\phi)}_{\sin\phi_0 + \Delta\phi\cos\phi_0}$$

$$= q_0\cos\phi_0 - q_0\Delta\phi\sin\phi_0 + \Delta q\cos\phi_0 - \Delta q\Delta\phi\sin\phi_0$$

$$- r_0\sin\phi_0 - r_0\Delta\phi\cos\phi_0 + \Delta r\sin\phi_0 + \Delta r\Delta\phi\cos\phi_0$$

Assume for regular flight

$$\phi_0 \approx 0, \quad q_0 = 0 = r_0$$

Simplify to

$$\Delta\dot{\theta} = \cancelto{0}{q_0\cos\phi_0} - \cancelto{0}{q_0\Delta\phi\sin\phi_0} + \Delta q\cancelto{1}{\cos\phi_0} - \cancelto{HOT}{\Delta q\Delta\phi\sin\phi_0}$$

$$- \cancelto{0}{r_0\sin\phi_0} - \cancelto{0}{r_0}\Delta\phi\cancelto{1}{\cos\phi_0} + \Delta r\cancelto{0}{\sin\phi_0} + \cancelto{0}{\Delta r\Delta\phi}\cancelto{1}{\cos\phi_0}$$

$$\boxed{\Delta\dot{\theta} = \Delta q}$$

So the previous result $I_y \Delta\dot{q} = \Delta M$ could be approximated as

$$I_y \frac{d}{dt}(\underbrace{\Delta q}_{\Delta\dot{\theta}}) = \Delta M$$

$$\boxed{I_y \ddot{\theta} = \Delta M}$$ This is what we used in physics and dynamics.

Where do Aerodynamics and Propulsion enter into the flight dynamics equations?

Body Frame forces and Moments

$$\underbrace{\underbrace{X, Y, Z}_{\substack{\text{lift, drag} \\ \text{side force}}} \qquad \underbrace{L, M, N}_{\substack{\text{pitch moment} \quad C_m \\ \cdots \qquad \cdots \\ \cdots \qquad \cdots}}}_{\text{Theses are a function of } u, v, w, \delta_e, \delta_a, \delta_T, \ldots}$$

The perturbations are also functions of states.

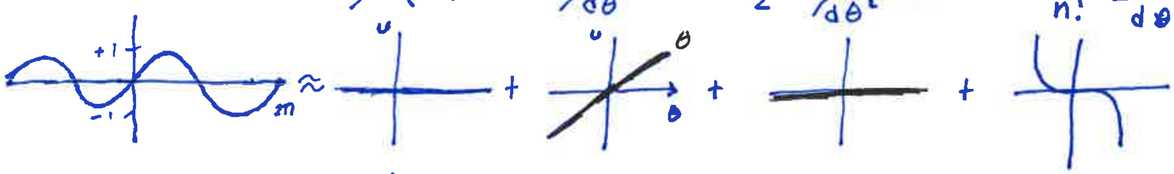
Taylor Series

Approximate ~~most~~ many functions with a series of derivative terms.

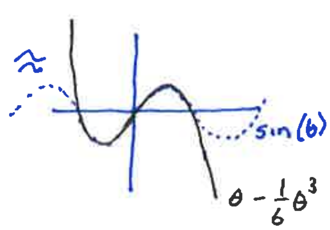
$$f(x_0 + \Delta x) = f(x_0) + \frac{df(x_0)}{dx}\Delta x + \frac{1}{2}\frac{d^2 f}{dx^2}(\Delta x)^2 + \cdots \frac{1}{n!}\frac{d^n f}{dx^n}(\Delta x)^n$$

Ex:

$$\sin(\overset{0+\theta}{\theta}) \approx \cancelto{0}{\sin(0)} + \cancelto{\cos(0) = 1}{\frac{d\sin(\theta)}{d\theta}}\,\theta + \frac{1}{2}\cancelto{-\sin(0) = 0}{\frac{d^2\sin(\theta)}{d\theta^2}}\,\theta^2 + \cdots \frac{1}{n!}\frac{d^n\sin(\theta)}{d\theta^n}(\theta)^n$$



$$0 + \theta + 0 + -\frac{1}{6}\theta^3$$



$$+\frac{1}{120}\theta^5 - \frac{1}{5040}\theta^7$$

within $-\frac{\pi}{3} < \theta < \frac{\pi}{3}$, the approximation $\theta - \frac{1}{6}\theta^3$ fits $\sin(\theta)$ fairly well. (Error 1%)

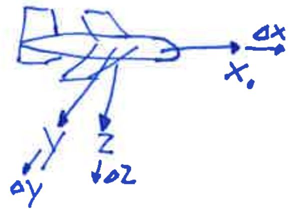
(warning: diverges outside this bound!)

We will approximate $\Delta X(u, v, w, \delta_e, \ldots)$ with a Taylor series (1^{st} order, linear)

$$\Delta X = \frac{\partial X}{\partial u}\Delta u + \frac{\partial X}{\partial w}\Delta w + \cdots$$

Which terms are the most important for ΔX, ΔY, ΔZ, ΔL, ΔM, ΔN?

$$\Delta X = \underbrace{\frac{\partial X}{\partial u}}_{\substack{\text{how does}\\ \text{the } x \text{ force}\\ \text{change with}\\ \text{velocity}\\ \text{perturbations}}} \Delta u + \underbrace{\frac{\partial X}{\partial w}} \Delta w + \underbrace{\frac{\partial X}{\partial \delta_e}}_{\substack{\text{how does the}\\ X \text{ force change}\\ \text{with elevator}\\ \text{deflection?}}} \Delta \delta_e + \underbrace{\frac{\partial X}{\partial \delta_T}}_{\substack{\text{how does the}\\ X \text{ force change}\\ \text{with thrust}\\ \text{setting.}\\ \text{"throttle"}}} \Delta \delta_T$$



$$\Delta Y = \frac{\partial Y}{\partial v}\Delta v + Y_p \Delta p + Y_r \Delta r + Y_{\delta_r}\Delta\delta_r$$

$$\Delta Z = Z_u \Delta u + Z_w \Delta w + Z_{\dot w}\Delta\dot w + Z_q \Delta q + Z_{\delta_e}\Delta\delta_e + Z_{\delta_T}\Delta\delta_T$$

$$\Delta L = \ldots$$

$$\Delta M = \ldots$$

$$\Delta N = \ldots$$

These terms (e.g. $\frac{dY}{dv} \equiv Y_v$) are <u>stability derivatives</u>.

Stability derivatives have units associated with them

Ex: $\frac{dY}{dv} = \left[\frac{\text{force}}{\text{velocity}}\right]$

The non-dimensional form is called a <u>stability coefficient</u>.

Ex: $\frac{dY}{dv} = \left[\frac{\text{force}}{\text{velocity}}\right]$ and $[\text{force}] = C_y\, q S$

$[\text{velocity}] = C_v\, u_0$

$$\Rightarrow Y_v = \frac{dC_y}{du}\, qS$$

$$= \frac{dC_y}{d(u/u_0)}\left(\frac{qS}{u_0}\right)$$

Stability Derivatives

Determine from:
- Prediction/Simulation,
- Wind Tunnel
- Flight Tests

We will look at Prediction tools here.
- Analytical
- DATCOM (compiled experimental data)
-

Ex:

$$\Delta X = \frac{\partial X}{\partial u}\Delta u$$

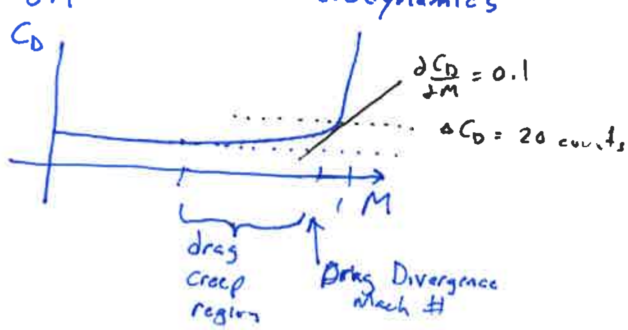
$$= \frac{\partial (T-D)}{\partial u}\Delta u = \frac{\partial T}{\partial u} - \frac{\partial D}{\partial u} = \frac{\partial T}{\partial u} - \frac{\partial}{\partial u}\left(\tfrac{1}{2}\rho u^2 S C_D\right)$$

C_D could be a function of u.

$$= \frac{\partial T}{\partial u} - \rho u_0 S C_D - \tfrac{1}{2}\rho u_0^2 S \frac{\partial C_D}{\partial u}$$

What is $\frac{\partial C_D}{\partial u}$? $\quad u = M \cdot a \quad \Rightarrow \quad \frac{\partial C_D}{\partial (Ma)} = \frac{1}{a}\frac{\partial C_D}{\partial M}$

We saw $\frac{\partial C_D}{\partial M}$ before in aerodynamics



$$C_{D_u} = \frac{\partial C_D}{\partial (u/u_0)} = u_0 \frac{\partial C_D}{\partial u} = \frac{u_0}{a}\frac{\partial C_D}{\partial M} = M\frac{\partial C_D}{\partial M}$$

Non-dimensionalization

$$[\text{Force}] = C_F \, q \, S$$

$$[\text{Angle}] = \alpha \quad (\text{radians are unitless})$$

$$[\text{Velocity}] = C_U \, U_0$$

$$[\text{Angular Rate}] = C_p \cdot \underbrace{\frac{2U_0}{b}}_{\text{roll derivative}} \quad \text{or} \quad C_r \underbrace{\frac{2U_0}{\bar{c}}}_{\text{pitch derivative}}$$

Book typos

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