Lesson 24

Longitudinal Motion

phugoid

Short period

Root Locus

Earlier, we saw that the solution to an ODE is written as

$$s(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \dots \; c_n e^{\lambda_n t}$$

where λ are the roots (or eigenvalues) of the system

$$\dot{s} = As$$

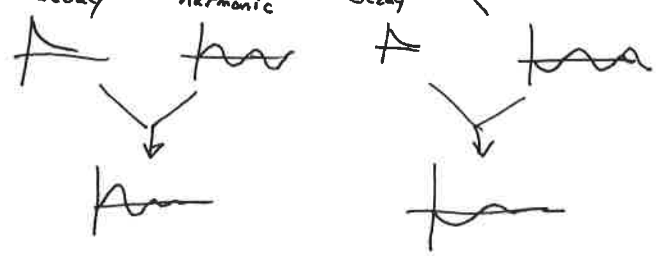
In general, the eigenvalues can be composed of both real and imaginary parts

$$\lambda = \eta \pm i\omega$$

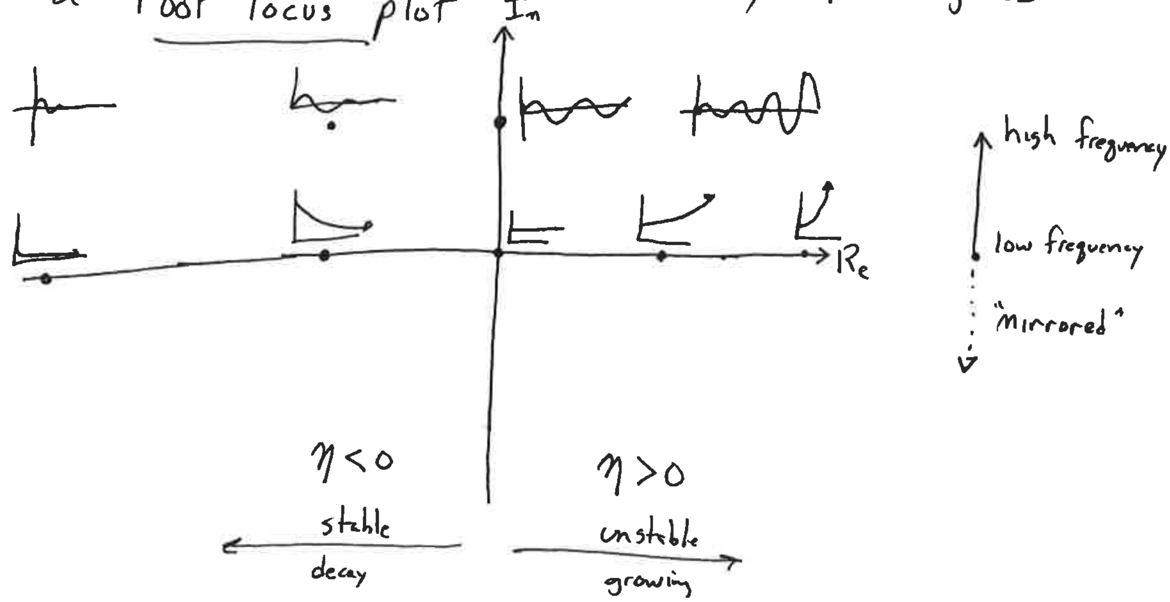
Ex: ~~2DOF~~ 2nd order system with $\lambda = -2 \pm i3$

$$s(t) = c_1 e^{(-2+i3)t} + c_2 e^{(-2-i3)t}$$

$$= c_1 \underbrace{e^{-2t}}_{\text{decay}} \underbrace{e^{i3t}}_{\text{harmonic}} + c_2 \underbrace{e^{-2t}}_{\text{decay}} e^{-i3t}$$



plotting η and ω on the real, imaginary plane gives a <u>root locus</u> plot



2^{nd} order ODE

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0 \quad \Rightarrow \quad \lambda = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{k}{m}}}{2}$$

The roots have a canonical form of

$$\lambda = \eta \pm i\omega$$

$$= -\xi\omega_n \pm i\omega_n\sqrt{1-\xi^2}$$

$$= \underbrace{-\xi\omega_n}_{\eta} \pm \sqrt{\underset{\substack{\uparrow \\ \text{natural freq with no damping}}}{-\omega_n^2} + \underbrace{\omega_n^2\xi^2}_{\eta^2}}$$

$\theta = a\cos\xi$

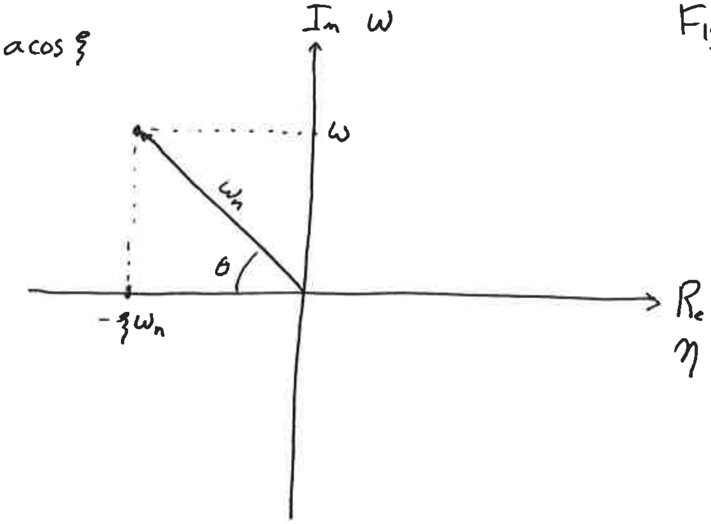
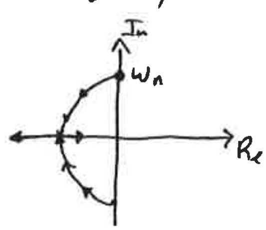


Figure 4.3

$$\omega_n = \omega^2 + \xi^2$$

For a given undamped natural frequency (i.e. same spring + mass) the damped frequency decreases with increased damping.



roots meet at the Re axis and join to give real roots

Writing the small-disturbance linearized longitudinal equations of motion in a system form gives

$$\dot{x} = [A]x + Bu$$

with the states $x = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix}$ Table 3.2

and controls $u = \begin{pmatrix} \Delta \delta_e \\ \Delta \delta_T \end{pmatrix}$

$$\begin{pmatrix} \Delta \dot{u} \\ \Delta \dot{w} \\ \Delta \dot{q} \\ \Delta \dot{\theta} \end{pmatrix} = \begin{bmatrix} X_u & X_w & 0 & -g \\ Z_u & Z_w & U_0 & 0 \\ M_u + M_{\dot{w}} Z_u & M_w + M_{\dot{w}} Z_w & M_q + M_{\dot{w}} U_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix}$$

$$+ \begin{bmatrix} X_{\delta_e} & X_{\delta_T} \\ Z_{\delta_e} & Z_{\delta_T} \\ M_{\delta_e} + M_{\dot{w}} Z_{\delta} & M_{\delta_T} + M_{\dot{w}} Z_{\delta_T} \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \Delta \delta_e \\ \Delta \delta_T \end{pmatrix}$$

$$= Ax + Bu$$

The eigenvalues of A give the damping and frequency.

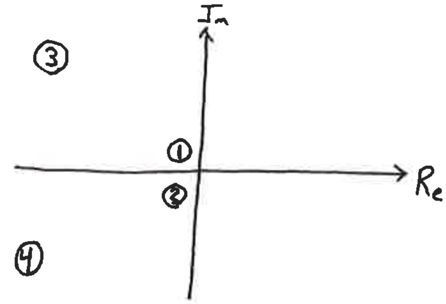
4 eigenvalues

See Table 4.2 for the derivative terms.

The longitudinal linearized equations gives 4 eigenvalues, but of two sets of conjugate pairs.

$$\lambda_{1,2} = -\text{small} \pm i \cdot \text{small} \quad \Big\} \text{ low damping, low frequency "phugoid"}$$

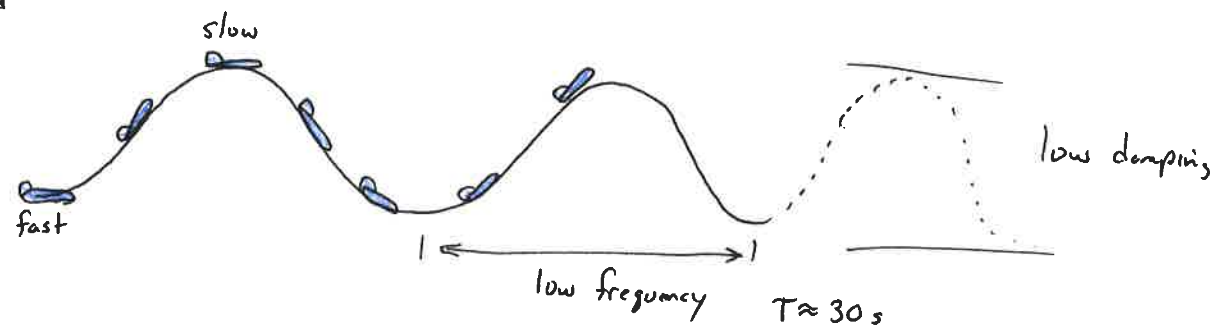
$$\lambda_{3,4} = -\text{large} \pm i \cdot \text{large} \quad \Big\} \text{ high damping, high frequency "short period"}$$



Looking at the eigenvectors shows that:

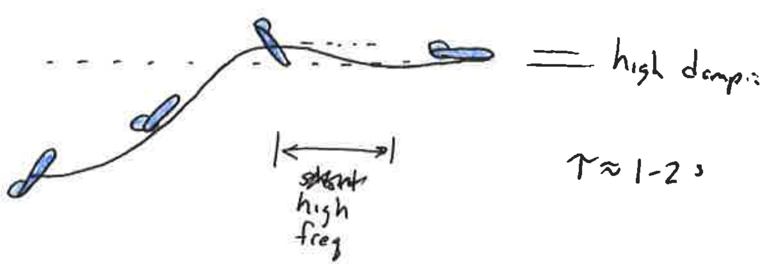
- The phugoid contains mostly ΔU and $\Delta\theta$
- The short period contains mostly $\Delta\alpha$ and $\Delta\theta$ $\quad \left(\Delta\alpha \approx \frac{\Delta W}{U_0} \right)$

Phugoid



Short Period

Exaggerated scale



very little velocity change

Approximation for Phugoid "few goyed"

The phugoid is essentially composed of Δu and $\Delta\theta$. There is very little Δw in other words $\Delta\alpha \approx \frac{\Delta w}{U_0} \approx 0$

Reducing the previous 4x4 gives

$$\begin{pmatrix} \Delta\dot u \\ \cancelto{0}{\Delta\dot w} \\ \Delta\dot q \\ \Delta\dot\theta \end{pmatrix} = \begin{bmatrix} X_u & X_w & 0 & -g \\ Z_u & Z_w & U_0 & 0 \\ \# & \# & \# & \# \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} \Delta u \\ \cancelto{0}{\Delta w} \\ \Delta q \\ \Delta\theta \end{pmatrix}$$

Thus, we can use the w row to solve for Δq!

$$0 = Z_u \Delta u + U_0 \Delta q \quad \Rightarrow \quad \Delta q = -\frac{Z_u \Delta u}{U_0}$$

Putting this back into the matrix gives

$$\begin{pmatrix} \Delta\dot u \\ \Delta\dot\theta \end{pmatrix} = \begin{bmatrix} X_u & -g \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \Delta u \\ \Delta\theta \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \Delta q \end{pmatrix}}$$

$$= \begin{pmatrix} 0 \\ -\frac{Z_u}{U_0} \end{pmatrix} \Delta u$$

$$= \begin{pmatrix} X_u & -g \\ -\frac{Z_u}{U_0} & 0 \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta\theta \end{pmatrix}$$

Find eigenvalues

$$\det(\lambda I - A) = 0 \quad \Rightarrow \quad \begin{vmatrix} \lambda - X_u & g \\ \frac{Z_u}{U_0} & \lambda \end{vmatrix} = \lambda^2 - \lambda X_u - g\frac{Z_u}{U_0} = 0$$

$$\lambda = \frac{X_u \pm \sqrt{X_u^2 + 4g\frac{Z_u}{U_0}}}{2}$$

We recognize $\lambda = -\zeta \omega_n \pm i\, \omega_n (1-\zeta^2)^{1/2}$

$$\omega_n = \sqrt{\frac{-Z_u\, g}{U_0}} \qquad \text{and} \qquad \zeta = -\frac{X_u}{2\omega_n}$$

From Table 4.2

$$X_u = \frac{-(C_{D_u} + 2C_{D_0})\, qS}{mU_0} \qquad Z_u = \frac{-(C_{L_u} + 2C_{L_0})\, qS}{mU_0}$$

For a subsonic aircraft, $C_{D_u} \approx 0$ and $C_{L_u} \approx 0$

$$X_u \approx \frac{-2C_{D_0}\, qS}{mU_0} \qquad Z_u \approx \frac{-2C_{L_0}\, qS}{mU_0} \qquad \text{and} \quad C_{L_0} \approx \frac{W}{qS}$$

Substitute

$$\omega_n \approx \sqrt{\frac{-\left(-2\frac{W}{qS}\right) qS}{mU_0}\,\frac{g}{U_0}} = \sqrt{\frac{2Wg}{mU_0^2}\,\frac{W}{mg}} = \frac{W}{mU_0}\sqrt{2} =$$

$$\boxed{\omega_n \approx \frac{g}{U_0}\sqrt{2}}$$

$$\zeta = +\frac{\cancel{2}C_{D_0}\cancel{qS}}{\cancel{m}\cancel{U_0}}\;\frac{\cancel{U_0}}{\cancel{2}\,g\sqrt{2}}\;\frac{\cancel{W}}{C_{L_0}\,\cancel{qS}}\;\frac{\cancel{m}g}{W}$$

$$= \frac{C_{D_0}}{C_{L_0}}\frac{1}{\sqrt{2}} = \boxed{\frac{1}{\sqrt{2}}\,\frac{1}{L/D} \approx \zeta}$$

Phugoid

- Frequency inversely related to velocity
- damping inversely related to L/D ratio

Efficient, high speed aircraft have persistent low frequency phugoid

History

Named by Lanchester who 1st found and characterized this motion.

He specifically looked for the Greek word for "flight".

Unfortunately, he used "φυγη" → phu which means flight as in "escape".

Short Period Approx

Assume $\Delta u = 0$

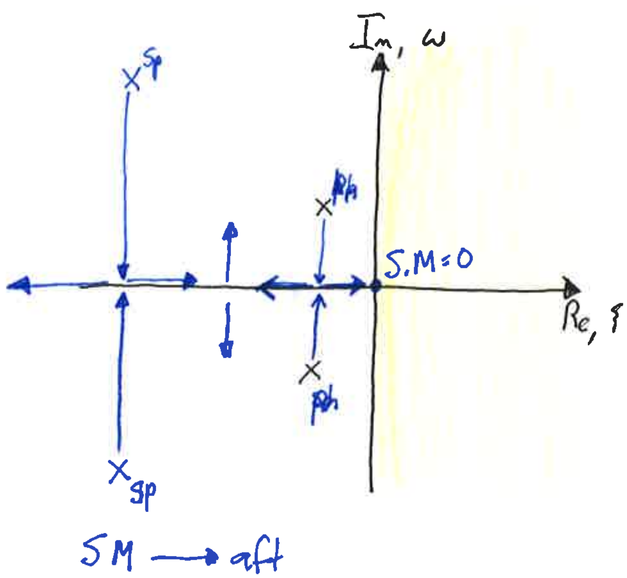
$$\begin{pmatrix} \Delta \dot{w} \\ \Delta \dot{q} \end{pmatrix} = \begin{pmatrix} Z_w & u_0 \\ M_w + M_{\dot{w}} Z_w & M_q + M_{\dot{w}} u_0 \end{pmatrix} \begin{pmatrix} \Delta w \\ \Delta q \end{pmatrix}$$

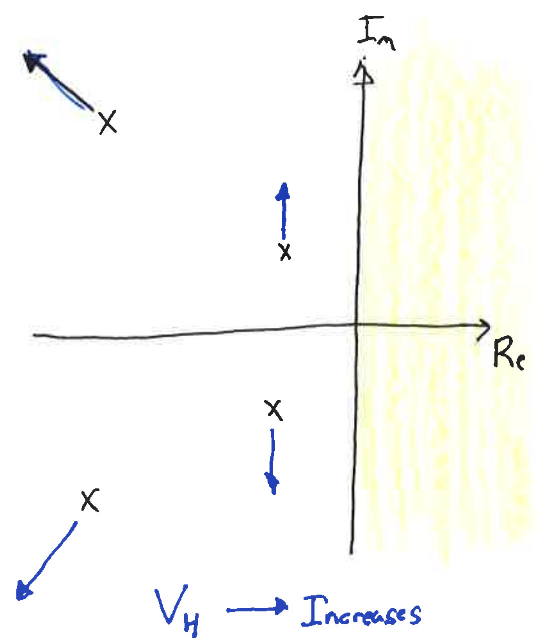
In a similar way

$$\boxed{\begin{aligned} \omega_n &= \sqrt{\frac{Z_\alpha M_q}{u_0} - M_\alpha} \\ \zeta &= \frac{-\left(M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0}\right)}{2\omega_n} \end{aligned}}$$

Substituting gives

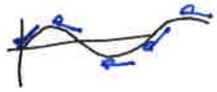
$$\omega_n \approx \sqrt{\left(\underbrace{\frac{-C_{L\alpha} C_{m_q} \bar{c}^2 S^2}{4 I_y} \frac{\rho}{m}}_{\text{Tail size} \Rightarrow C_{m_q} \text{ and density ratio } \rho/m} - \underbrace{\frac{C_{m_\alpha} S \bar{c}}{I_y}}_{\text{Static Margin in } C_{m\alpha}}\right) q}$$

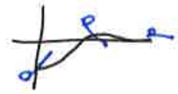




Summary

- The dynamics of longitudinal motion (pitching motion, velocity) indicates two behaviors: ① Phugoid and ② Short Period

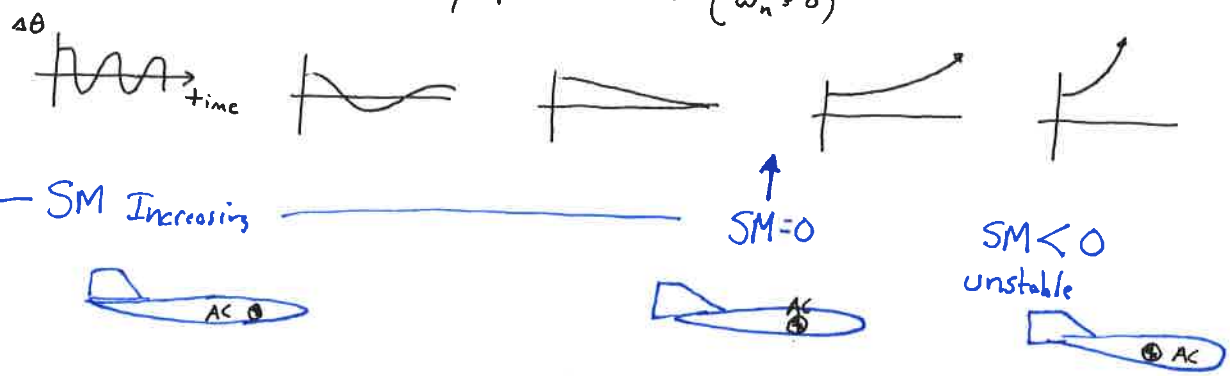




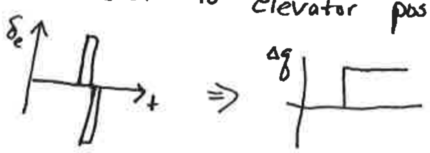
- Phugoid scales with $\omega = \frac{g}{U_0}\sqrt{2}$ and $\zeta = \frac{1}{\sqrt{2}} \frac{1}{L/D}$

- Short Period scales with Tail size and Static margin

- At a static margin of zero, the phugoid damping ($\lambda = \eta \pm i\omega$) goes through zero. The imaginary part is zero ($\omega_n = 0$)



- The short period is often felt in turbulence when Δw changes due to atmospheric disturbances. We can also excite S.P through Δq with a doublet to elevator position



without Δu

- Excessively stable aircraft (SM too large) are overly damped in rough air. The ride quality is poor.