

Lesson 5

Vorticity

26th Aug '15

- Homework due 2nd Sept

- Readings

FVA chap 2

Vorticity Video <http://tiny.cc/VorticityFilm>

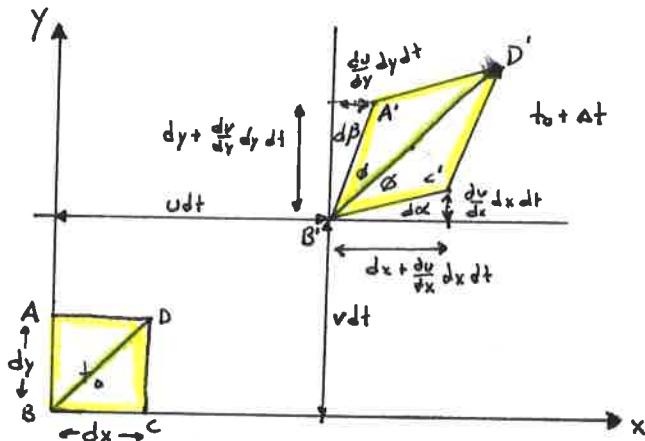
Vorticity

Definition $\omega = \nabla \times V$
 $= \text{curl } V$

in 3D cartesian coordinates,

$$\omega = \left(\frac{du}{dy} - \frac{dv}{dz} \right) \hat{i} - \left(\frac{dw}{dx} - \frac{du}{dz} \right) \hat{j} + \left(\frac{dv}{dx} - \frac{dw}{dy} \right) \hat{k}$$

Distortion of a fluid element ABCD to A'B'C'D'



Define rotation as the angle of line
 $BD \rightarrow B'D'$

$$d\Omega = \underbrace{\phi + d\alpha}_{\text{final}} - \underbrace{\frac{\pi}{4}}_{\text{initial}}$$

and

$$2\phi + d\beta + d\alpha = \frac{\pi}{2} \Rightarrow \phi = \frac{\pi}{4} - \frac{d\beta}{2} - \frac{d\alpha}{2}$$

$$\begin{aligned} d\Omega &= \frac{\pi}{4} - \frac{d\beta}{2} - \frac{d\alpha}{2} + d\alpha - \frac{\pi}{4} \\ &= \frac{d\alpha}{2} - \frac{d\beta}{2} \end{aligned}$$

Angles

$$d\beta = \arctan \left(\frac{\frac{du}{dy} dy dt}{dy + \frac{dv}{dy} dy dt} \right) \quad \text{as } dt \approx 0 \quad (\text{formally, } \lim_{dt \rightarrow 0})$$

$$\approx \frac{\frac{du}{dy} dy dt}{dy + \frac{dv}{dy} dy dt} \cdot \frac{\frac{1}{dy dt}}{\frac{1}{dy dt}} = \frac{\frac{du}{dy}}{\frac{1}{dt} + \frac{dv}{dy}}$$

when $dt \rightarrow 0$, $\frac{1}{dt} \gg \frac{dv}{dy} \Rightarrow \frac{du}{dy} dt$

$$d\alpha = \text{same strategy. } \approx \frac{dv}{dx} dt$$

Rotation

$$d\Omega = \frac{d\alpha}{2} - \frac{d\beta}{2} = \frac{1}{2} \frac{dv}{dx} dt - \frac{1}{2} \frac{du}{dy} dt$$

Vorticity

$$\omega \equiv 2 \frac{d\Omega}{dt} = \frac{dv}{dx} - \frac{du}{dy} = \underbrace{\nabla \times V}_{2D}$$

Vorticity is defined as twice the angular velocity

$\omega = \nabla \times V$

Vorticity Transport

$$\omega = \nabla \times V$$

- Start with momentum equation

$$\frac{\partial V}{\partial t} + V \cdot \nabla V = f - \frac{\nabla P}{\rho} + \frac{\nabla \cdot \bar{\tau}}{\rho}$$

Rewrite in terms of $\omega \Rightarrow V \cdot \nabla V = \frac{1}{2} \nabla(V \cdot V) - V \times \omega$

- Apply curl operator to entire equation ($\nabla \times (\dots)$)

$$\underbrace{\nabla \times \frac{\partial V}{\partial t}}_{\frac{\partial}{\partial t}(\nabla \times V)} + \frac{1}{2} \nabla \times (\nabla(V \cdot V)) - \nabla \times (V \times \omega) = \nabla \times f - \nabla \times \left(\frac{\nabla P}{\rho} \right) + \nabla \times \left(\frac{\nabla \cdot \bar{\tau}}{\rho} \right)$$

f is irrot
"think gravity"

- Time derivative of ω appears.

$$\frac{\partial \omega}{\partial t} - \underbrace{\nabla \times (V \times \omega)}_{= -} = - \nabla \times \left(\frac{\nabla P}{\rho} \right) + \nabla \times \left(\frac{\nabla \cdot \bar{\tau}}{\rho} \right)$$

- $\nabla \times (V \times \omega) = \underbrace{V \nabla \cdot \omega}_{\nabla \cdot (V \times V)} - \omega \nabla \cdot V + \omega \cdot \nabla V - V \cdot \nabla \omega$

- $\nabla \times \left(\frac{\nabla P}{\rho} \right)$ Identity says $\nabla \times (c A) = c (\nabla \times A) + (\nabla c) \times A$
thus $\nabla \times \left(\tilde{\rho}^{-1} \nabla P \right) = \tilde{\rho}^{-1} (\nabla \times \nabla P) + (\nabla \tilde{\rho}^{-1}) \times \nabla P$

and $\nabla \tilde{\rho}^{-1} = \nabla \left(\frac{f}{g} \right) = g \frac{\nabla f - f \nabla g}{g^2} = \frac{f \nabla \vec{r} - 1 \nabla f}{\rho^2}$

$$\nabla \times \left(\frac{\nabla P}{\rho} \right) = - \frac{\nabla \rho \times \nabla P}{\rho^2}$$

- $\nabla \times \left(\frac{\nabla \cdot \bar{\tau}}{\rho} \right)$ leave it alone.

$$\frac{\partial \omega}{\partial t} + \omega \nabla \cdot V - \omega \cdot \nabla V + V \cdot \nabla \omega = \frac{\nabla f \times \nabla P}{\rho^2} + \nabla \times \left(\frac{\nabla \cdot \bar{\tau}}{\rho} \right)$$

a

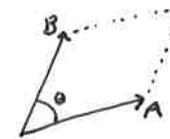
Eulerian frame $\frac{\partial \omega}{\partial t} - \nabla \times (V \times \omega) = \frac{\nabla \rho \times \nabla P}{\rho^2} + \nabla \times \left(\frac{\nabla \cdot \bar{\tau}}{\rho} \right)$

Lagrangian frame $\frac{D \omega}{Dt} \left(\frac{\omega}{\rho} \right) = \frac{\omega}{\rho} \cdot \nabla V + \frac{\nabla \rho \times \nabla P}{\rho^3} + \frac{1}{\rho} \nabla \times \left(\frac{\nabla \cdot \bar{\tau}}{\rho} \right)$

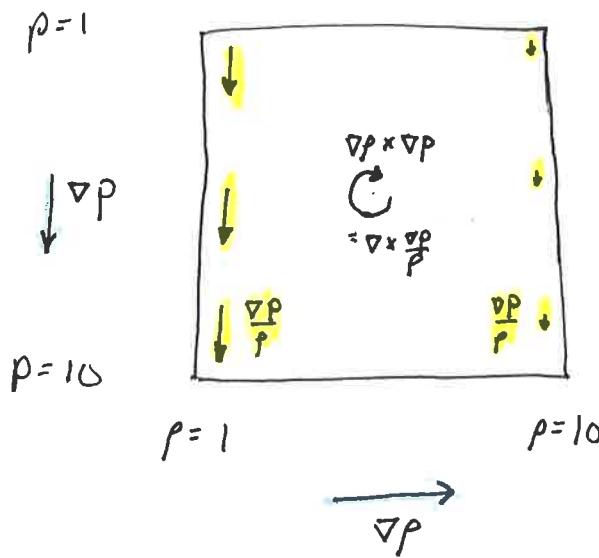
Baroclinic

$$\nabla p \times \nabla p$$

Remember $A \times B = |A||B| \sin \theta$



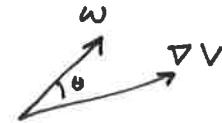
If the density and pressure gradients are not aligned, vorticity is generated.



$$\nabla p \times \nabla p \propto -\nabla \times \left(\frac{\nabla p}{\rho} \right)$$

Vortex Stretching

$$\omega \cdot \nabla V = |\omega| |\nabla V| \cos \theta$$



Conservation of angular momentum.

Vorticity increases when the vorticity vector is aligned with accelerating flow.

Ex: Certain B737 models have a cowl mounted vortex generator located such that the circulation travels over the wing. Where is $|\omega|$ greatest? Assume subsonic flow.



$|\omega|$ greatest where $\frac{d}{dr} (|\omega| |\nabla V| \cos \theta) = 0$

$$A: \nabla V \leftarrow \Rightarrow \frac{d\omega}{dr} < 0$$

$$B: \nabla V \rightarrow \Rightarrow \frac{d\omega}{dr} > 0$$

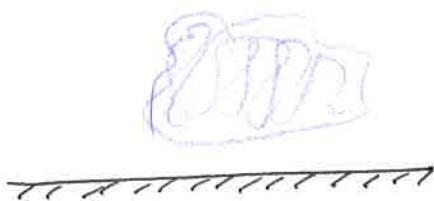
$$C: \nabla V \approx 0 \Rightarrow \frac{d\omega}{dr} \approx 0$$

$$D: \nabla V \approx 0 \Rightarrow \frac{d\omega}{dr} \approx 0$$

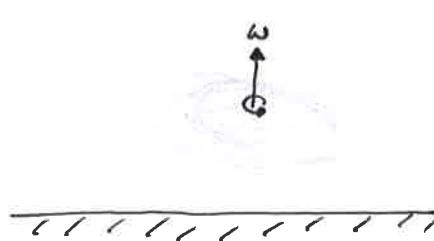
C

Vortex Stretching

- Say we have an airmass



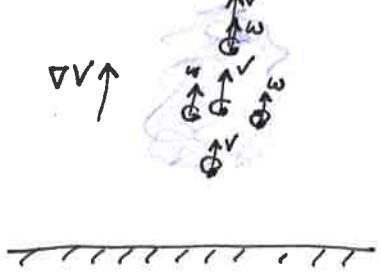
- Through some process, it ~~now~~ now has vorticity (of a low magnitude)



perhaps $\nabla p \times \nabla p \neq 0$

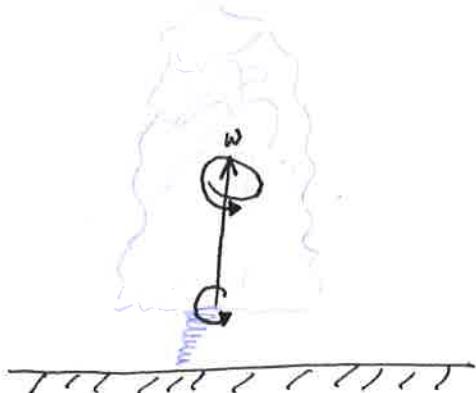
or
 ∇p and ∇p are not aligned.
in the boundary layer

- Given that the airmass is unstable (hot humid air), the air accelerates upward



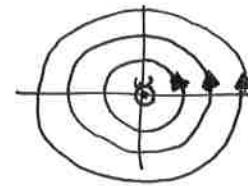
$$\begin{aligned}\frac{Dw}{Dt} &= \omega \cdot \nabla V + v \cancel{\partial}^{\text{small}} w \\ &= |w| |DV| \cos \theta\end{aligned}$$

- Vortex stretching increases the vorticity



Given a region of vorticity where circular streamlines exist ~~and no shear motion~~
such that $\omega \cdot \nabla V = 0$, the governing equation is

$$\frac{D\omega}{Dt} = \omega \cancel{\nabla V}^{\circ} + \nu \nabla^2 \omega$$



ω is a vector out of the page.

How does the vorticity behave?

$$\frac{D\omega}{Dt} = \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) = \nu \left(\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r} \frac{\partial \omega}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \omega}{\partial \theta^2} \right)$$

This is a PDE with a classic Separation of Variables approach.

$$\omega = T R \Theta$$

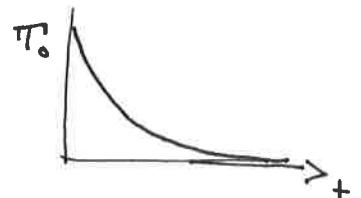
Subst.

$$\frac{T_+}{T} = \nu \left(\frac{TR_{rr}}{R} + \frac{1}{r} \frac{TR_r}{R} + \frac{1}{r^2} \frac{\Theta_{\infty}}{\Theta} \right) = -\lambda$$

Thus, in time

$$\frac{1}{\nu} \frac{dT}{dt} = -\lambda T \Rightarrow \frac{dT}{dt} + \nu \lambda T = 0$$

$$\text{Solution is } T = T_0 e^{-\nu \lambda t}$$



Vorticity decays to zero exponentially (eventually)

Isentropic

$$T \frac{Ds}{Dt} = (\hat{\tau} \cdot \nabla) \cdot V - \nabla \cdot \dot{g} + \dot{g}_v$$

↑ Viscous ↑ wall heat flux ↑ internal heating

Isentropic Ideal Gas

$$\frac{dp}{P} = \gamma \frac{dp}{\rho} = \frac{\gamma}{\sigma-1} \frac{dh}{h}$$

thus ↑ enthalpy
not altitude...

$$\frac{P_2}{P_1} = \left(\frac{P_2}{P_1} \right)^\gamma = \left(\frac{h_2}{h_1} \right)^{\frac{\gamma}{\sigma-1}}$$

Soln:

$$\int \frac{dp}{P} = \ln P$$

$$\gamma \int \frac{dr}{P} = \gamma \ln P \Rightarrow \ln P_2^{\frac{1}{\gamma}} = \gamma \ln P_1^{\frac{1}{\gamma}}$$

$$\ln P_2 / \ln P_1 = \gamma (\ln P_2 - \ln P_1)$$

$$\ln \left(\frac{P_2}{P_1} \right) = \gamma \ln \left(\frac{P_2}{P_1} \right)^\gamma$$

exp to both sides

$$\frac{P_2}{P_1} = \left(\frac{P_2}{P_1} \right)^\gamma$$

Where is this valid? (see FVA p14)

