

Lesson 6

28<sup>th</sup> Aug 2015

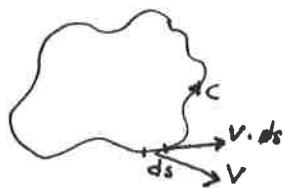
Circulation

# Circulation $\Gamma$

Definition

$$\Gamma = - \oint_C \mathbf{V} \cdot d\mathbf{s}$$

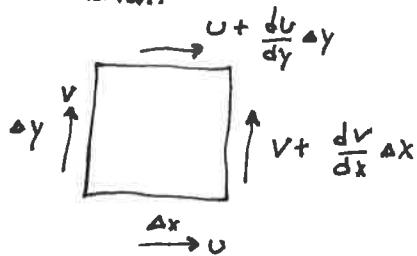
English



Circulation is proportional to the velocity component tangent to a closed curve.

Related to vorticity...

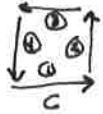
- Take a small element



- Definition of  $\Gamma$  with a path  $C$  as the edges of the element (ccw)

$$\begin{aligned} \Delta \Gamma &= - \oint_C \mathbf{V} \cdot d\mathbf{s} = - \int_1 V \cdot ds + - \int_2 V \cdot ds + \dots + \int_4 V \cdot ds \\ &= - \int_1 u \Delta x - \int_2 \left(v + \frac{dv}{dx} \Delta x\right) \Delta y - \int_3 \left(u + \frac{du}{dy} \Delta y\right) (\Delta x)(-1) - \int_4 v \Delta y (-1) \end{aligned}$$

all terms are constant

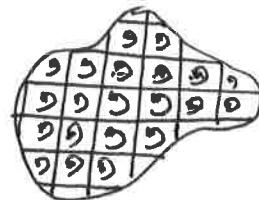
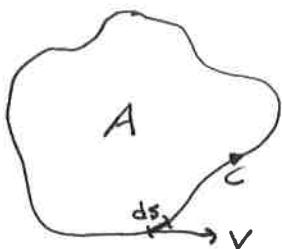


$$= -u \Delta x - v \Delta y - \frac{dv}{dx} \Delta x \Delta y + u \Delta x + \frac{du}{dy} \Delta y \Delta x + v \Delta y$$

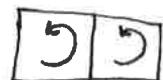
$$\Delta \Gamma = \underbrace{\left( \frac{du}{dy} - \frac{dv}{dx} \right)}_{\nabla \times \mathbf{V}} \Delta x \Delta y$$

This is  $\nabla \times \mathbf{V}$  in 2D !!

- Expand to region using summation of areas



since



↑  
integral  
along neighbor  
cancel.

$$\oint \mathbf{V} \cdot d\mathbf{s} = \sum \Delta \Gamma = \lesssim \omega \text{ area}$$

$$\boxed{\Gamma = - \oint \mathbf{V} \cdot d\mathbf{s} = - \iint_A \boldsymbol{\omega} \cdot \hat{\mathbf{n}} dA}$$

Circulation is the integral of vorticity.

## Irrational Flow

When  $\omega = \nabla \times V = 0$ , the flow is irrational.

Irrational flows have zero circulation since  $\Gamma = -\iint_A \vec{V} \cdot \vec{n} dA = 0$

Also, the term  $\int_C V \cdot ds$  is independent of path, since  $\Gamma = -\oint_C V \cdot ds$

Ex:  $\int_C V \cdot ds = 0$  when  $\nabla \times V = 0$ . Is the open curve D always; even if integral zero?

$$\int_D V \cdot ds \stackrel{?}{=} 0 \quad \text{where} \quad \begin{array}{c} \curvearrowright \\ D \end{array}$$

No, only a closed curve in an irrational flow is always zero.

From math, if  $\int_C V \cdot ds = 0$  regardless of path, then  $V \cdot ds = d\phi$   
where  $\phi$  is a scalar field of unique value (i.e. function).

$$\begin{aligned} \text{Expand } V \cdot ds = d\phi \quad \text{where } ds = (dx, dy, dz) \quad \text{and } d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \\ u dx + v dy + w dz = \underbrace{\frac{\partial \phi}{\partial x} dx}_{\text{These terms match!}} + \underbrace{\frac{\partial \phi}{\partial y} dy}_{u = \frac{\partial \phi}{\partial x}} + \underbrace{\frac{\partial \phi}{\partial z} dz}_{v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}} \end{aligned}$$

$$\text{In general, } V = \nabla \phi$$

We call  $\nabla \phi$  the velocity potential. Notice that  $\omega = \nabla \times V$

$$= \nabla \times (\nabla \phi) = 0$$

All irrational flows have a corresponding potential field. Thus, an irrational flow is often called a potential flow.

If vorticity exists in a flow, the  $\phi$  field is no longer single valued.

If incompressible,  $\nabla \cdot V = 0$  thus  $\nabla \cdot \nabla \phi = 0$  thus  $\nabla^2 \phi = 0$  (more later!)

# Kelvin's Theorem

Lord Kelvin: 1824 - 1907

Temperature unit  
named after him

He took the name Kelvin after the river Kelvin flowing through Scotland.

Pick a curve (closed) along a closed fluid path



Take time derivative of this circulation

$$-\frac{d\Gamma}{dt} = \frac{d}{dt} \left( \oint_C \mathbf{V} \cdot d\mathbf{s} \right) \Rightarrow \text{chain rule} \Rightarrow -\frac{d\Gamma}{dt} = \oint_C \frac{dV}{dt} \cdot d\mathbf{s} + \oint_C \mathbf{V} \cdot \frac{d}{dt}(d\mathbf{s})$$

The momentum eqn for an inviscid flow is

$$\frac{dV}{dt} = \mathbf{f} - \frac{1}{\rho} \nabla p \quad \text{when } \mathbf{f} = -\nabla F \quad (\text{i.e. irrotational force}) \quad \frac{dV}{dt} = \nabla F - \frac{1}{\rho} \nabla p$$

Also, here,

$$\frac{d}{dt}(d\mathbf{s}) = d\left(\frac{ds}{dt}\right) = d(V) = dV$$

Substitute:

$$-\frac{d\Gamma}{dt} = \oint_C \nabla F \cdot d\mathbf{s} - \oint_C \frac{1}{\rho} \nabla p \cdot d\mathbf{s} + \oint_C \mathbf{V} \cdot dV$$

$$-\frac{d\Gamma}{dt} = \underbrace{\oint_C \mathbf{f} \cdot d\mathbf{s}}_{0 \text{ when } \mathbf{f} \text{ is conservative (gravity)}} - \underbrace{\oint_C \frac{1}{\rho} \nabla p \cdot d\mathbf{s}}_{\text{Exact differential when } \rho = f(p)} + \underbrace{\oint_C \mathbf{V} \cdot dV}_{\text{Exact differential}}$$

$$\nabla \cdot \mathbf{V} = u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial w}{\partial z}$$

over a closed curve = 0

$$= 0$$

In a flow field without viscosity, a conservative body force, and a barotropic density, the circulation does not change in time.

$$\boxed{\frac{d\Gamma}{dt} = 0}$$

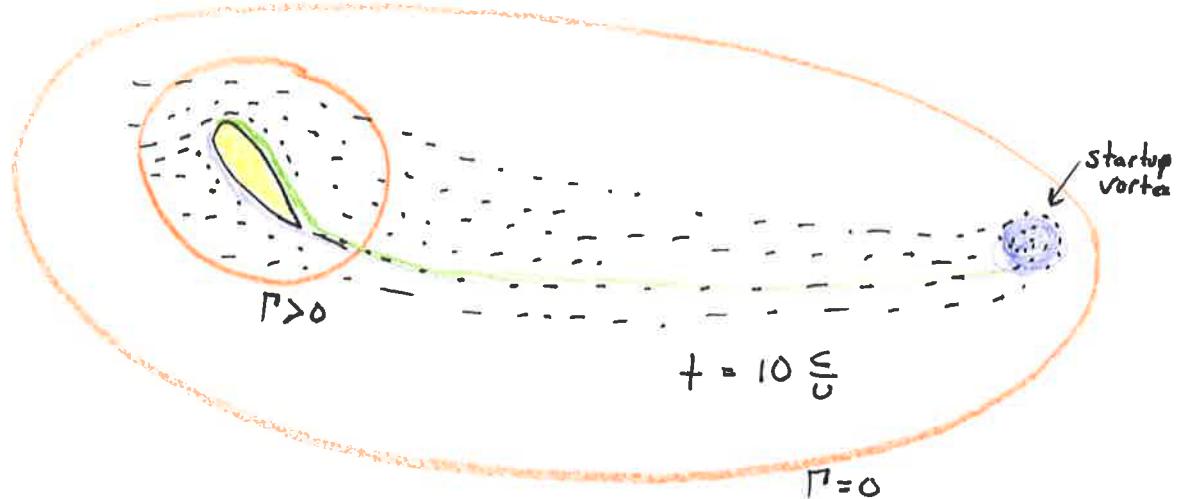
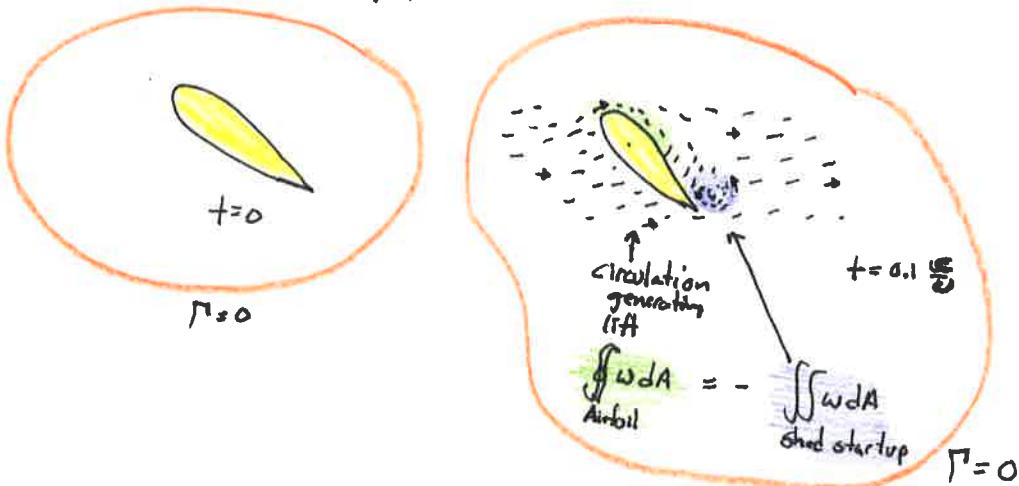
Nothing was restricted regarding  $\omega$  (other than inviscid flow....), so this is valid for the pressure field (i.e. Euler terms) of the governing equations.

## Kelvin's Theorem. (continued)

If  $\frac{d\Gamma}{dt} = 0$ , then  $\Gamma$  in a flow is constant.

Thus, any inviscid vorticity generated must have an equal and opposite circulation elsewhere.

Ex. Impulsive start of an airfoil



## Duality of Velocity and Vorticity - Dilatation (aka. Vorticity - Source)

By now, you've seen that vorticity and dilatation are just operations on velocity.  $\omega$  and  $\sigma$  are just special ways to look at velocity.

$$\omega = \nabla \times V \quad \sigma = \nabla \cdot V$$

What about the inverse,

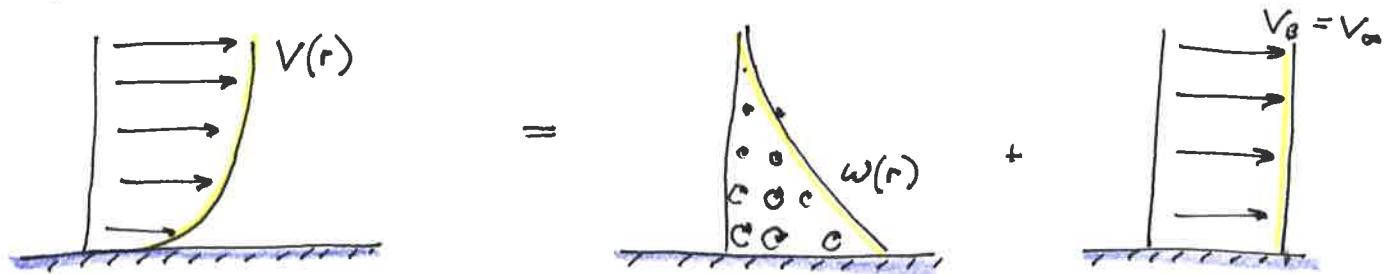
$$V = V_\sigma + V_\omega + V_b$$

$$V_\sigma = \frac{1}{4\pi} \iiint \sigma(r') \frac{r-r'}{(r-r')^3} dx' dy' dz'$$

$$V_\omega = \frac{1}{4\pi} \iiint \omega(r') \times \frac{r-r'}{(r-r')^3} dx' dy' dz'$$

$$V_b = V_\infty$$

Ex: Boundary Layer



No approximation. This is exactly dual.



XFOIL is an interactive program for the design and analysis of subsonic isolated airfoils.

It consists of a collection of menu-driven routines which perform various useful functions such as:

- Viscous (or inviscid) analysis of an existing airfoil, allowing
  - forced or free transition
  - transitional separation bubbles
  - limited trailing edge separation
  - lift and drag predictions just beyond CL<sub>max</sub>
  - Karman-Tsien compressibility correction
  - fixed or varying Reynolds and/or Mach numbers
- Airfoil design and redesign by interactive modification of surface speed distributions, in two methods:
  - Full-Inverse method, based on a complex-mapping formulation
  - Mixed-Inverse method, an extension of XFOIL's basic panel method
- Airfoil redesign by interactive modification of geometric parameters such as
  - max thickness and camber, highpoint position
  - LE radius, TE thickness
  - camber line via geometry specification
  - camber line via loading change specification
  - flap deflection
  - explicit contour geometry (via screen cursor)
- Blending of airfoils
- Writing and reading of airfoil coordinates and polar save files
- Plotting of geometry, pressure distributions, and multiple polars

## Release Conditions

Dilatation Transport  $\sigma = \nabla \cdot V$

- Start with momentum eqn.

$$\underbrace{\frac{\partial V}{\partial t} + V \cdot \nabla V}_{\frac{DV}{Dt}} = f - \frac{\nabla P}{\rho} + \frac{\nabla \cdot \bar{\tau}}{\rho}$$

- Apply div

$$\underbrace{\nabla \cdot \frac{DV}{Dt}}_{\frac{D}{Dt}(\nabla \cdot V)} = \nabla \cdot f - \nabla \cdot \left( \frac{\nabla P}{\rho} \right) + \nabla \cdot \left( \frac{\nabla \cdot \bar{\tau}}{\rho} \right)$$

For a constant density flow (incompressible)

$$\frac{D\sigma}{Dt} = \nabla \cdot f - \frac{1}{\rho} \nabla^2 P + \left( \frac{d^2 \gamma_{xx}}{dx^2} + 2 \frac{d^2 \gamma_{xy}}{dx dy} + 2 \frac{d^2 \gamma_{xz}}{dx dz} + 2 \frac{d^2 \gamma_{yz}}{dy dz} \right)$$

# Lagrangian and Eulerian frames

$$\frac{D(\cdot)}{Dt} = \frac{\partial(\cdot)}{\partial t} + \mathbf{V} \cdot \nabla(\cdot) = \underbrace{\frac{\partial(\cdot)}{\partial t}}_{\substack{\uparrow \\ \text{particle} \\ \text{moving} \\ \text{with } \mathbf{V}}} + \underbrace{\frac{\partial(\cdot)}{\partial x_i} \frac{dx_i}{dt}}_{\substack{\uparrow \\ \text{fixed} \\ \text{location} \\ \uparrow \\ \text{convective} \\ \text{terms}}} = \frac{\partial(\cdot)}{\partial t} + \underbrace{\frac{\partial(\cdot)}{\partial x_i} \frac{dx_i}{dt}}_{\substack{\text{just the} \\ \text{chain rule.}}}$$

Ex:

If the lapse rate in the atmosphere is  $-5^\circ$  per 1000 ft and the aircraft climbs at 1500 ft/min, what is the rate of change in temperature as the a/c climbs?

seen by the pilot

Need  $\frac{DT}{Dt}$  = Rate of change of temp experienced by particle/observer moving at velocity  $\mathbf{V}$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + 1500 \frac{\text{ft}}{\text{min}} \cdot \frac{\partial T}{\partial h} = 1500 \frac{\text{ft}}{\text{min}} \cdot \frac{-5^\circ}{1000 \text{ft}} = -7.5 \frac{^\circ}{\text{min}}$$

Ex:

Now, the atmosphere has a lapse rate of  $-5^\circ$  per 1000 ft, but is cooling at  $1^\circ$  per minute.

$$T(h,t) = T_0 - 5 \cdot \frac{h}{1000 \text{ft}} - 1 \cdot \frac{t}{\text{min}}$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \mathbf{V} \cdot \frac{\partial T}{\partial h} = -1 \frac{^\circ}{\text{min}} \quad 1500 \frac{\text{ft}}{\text{min}} \left| \frac{-5^\circ}{1000 \text{ft}} \right. = -8.5 \frac{^\circ}{\text{min}}$$