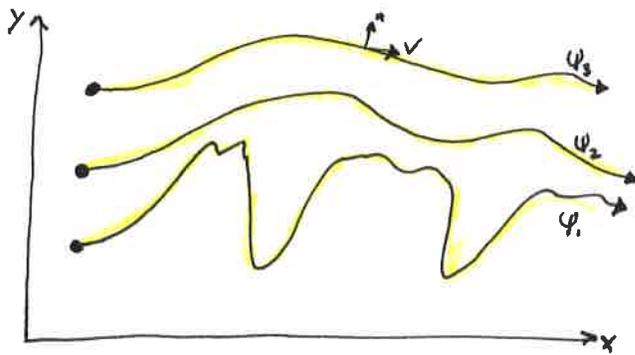


Lesson 7

Joukowski Airfoil

31st Aug 2015

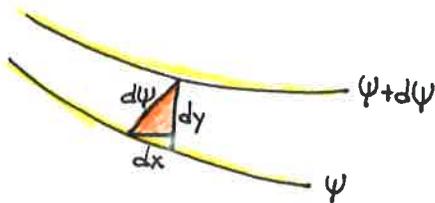
Stream function



Streamlines are defined as lines which no fluid crosses. ($\mathbf{v} \cdot \mathbf{n} = 0$)

The difference in magnitude between two stream functions provides a measure of the volume of fluid flowing between the two streamlines per second (per unit depth).
 ψ [ft^3/s]

Pick a control volume between two streamlines (ψ and $\psi + d\psi$)



$$\text{Flow into CV} = d\psi$$

$$\text{Flow out of CV} = u dy - v dx$$

↑ negative since normal for dx side is $-\hat{j}$

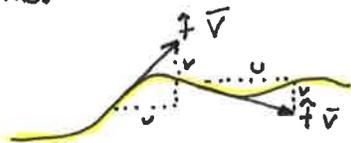
Definition of an exact differential of ψ

$$d\psi = \frac{d\psi}{dx} dx + \frac{d\psi}{dy} dy$$

Match terms

$$\frac{d\psi}{dx} dx + \frac{d\psi}{dy} dy = d\psi = u dy - v dx \quad \Rightarrow \quad u = \frac{d\psi}{dy} \quad \text{and} \quad v = -\frac{d\psi}{dx}$$

Streamlines



"The streamline is a curve whose tangent at every point coincides with the direction of the velocity vector"

Bertin-Smidt

$$\frac{dx}{dy} = \frac{u}{v} \quad \Rightarrow \quad \underbrace{u dy - v dx}_{= d\psi} = 0$$

we saw this term above

Thus along a streamline, no flow crosses the curve.

Div and Curl (incompressible, irrotational)

$$\nabla \cdot \mathbf{v} \stackrel{?}{=} 0 \quad \Rightarrow \quad \frac{du}{dx} + \frac{dv}{dy} = \frac{d^2\psi}{dx dy} - \frac{d^2\psi}{dy dx} = 0 \quad \checkmark$$

$$\nabla \times \mathbf{v} \stackrel{?}{=} 0 \quad \Rightarrow \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ u & v & w \end{vmatrix} = \frac{dv}{dx} - \frac{du}{dy} = \frac{d}{dx} \left(-\frac{d\psi}{dx} \right) - \frac{d}{dy} \left(\frac{d\psi}{dy} \right) = \nabla^2 \psi = 0$$

ψ is also a Laplacian for incomp + irrot.

In fact, $F = \phi + i\psi$

$$\nabla^2 F = 0$$

Joukowski Transform (circle to airfoil and back)

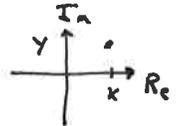
We want  but potential flow theory gives us circles 

Remember that both ϕ and ψ were shown to satisfy $\nabla^2 F = 0$ where $F = \phi + i\psi$

Conformal Transforms with $\nabla^2 F = 0$

Complex theory shows that if $\nabla^2 F = 0$, then any analytic function mapping also has $\nabla^2 F = 0$

Ex: $f(z) = f(x+iy) \Rightarrow z = x+iy$



$$\frac{df}{dx} = \frac{df}{dz} \frac{dz}{dx} = \frac{df}{dz} \cdot 1 = \frac{df}{dz} \quad \text{and} \quad \frac{df}{dy} = \frac{df}{dz} \frac{dz}{dy} = i \frac{df}{dz}$$

$$\frac{d}{dx} \left(\frac{df}{dx} \right) = \frac{d}{dz} \left(\frac{df}{dz} \right) \frac{dz}{dx} = \frac{d}{dz} \left(\frac{df}{dz} \right) = \frac{d^2 f}{dz^2}$$

$$\frac{d}{dy} \left(\frac{df}{dy} \right) = \frac{d}{dz} \left(\frac{df}{dy} \right) \frac{dz}{dy} = \frac{d}{dz} \left(i \frac{df}{dz} \right) i = i^2 \frac{d^2 f}{dz^2} = -\frac{d^2 f}{dz^2}$$

Apply to Laplace eqn

$$\nabla^2 F = 0 = \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} = \frac{d^2 f}{dz^2} - \frac{d^2 f}{dz^2} = 0 \quad \checkmark$$

In fact, any analytic function mapping still satisfies $\nabla^2 = 0$
 $z = (x+iy)^n$

Q: Can we find a complex function that transforms a circle to an airfoil?

Yes

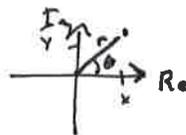
and it is simple

$$\boxed{J(z) = z + \frac{c_1^2}{z}}$$

Joukowski Transform

And in fact, there are many discovered mapping functions

Joukowski (cont) $J(z) = z + \frac{c_1^2}{z}$



Remember the properties of complex #s

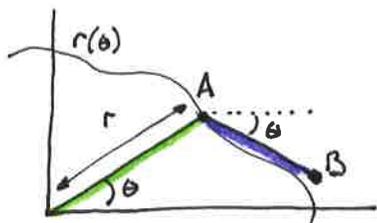
$$z = x + iy = r e^{i\theta}$$

$$\text{thus } \frac{1}{z} = \frac{1}{|z|} e^{-i\theta} = \frac{1}{r} e^{-i\theta}$$

The map is $J(z) = z + c_1^2/z$

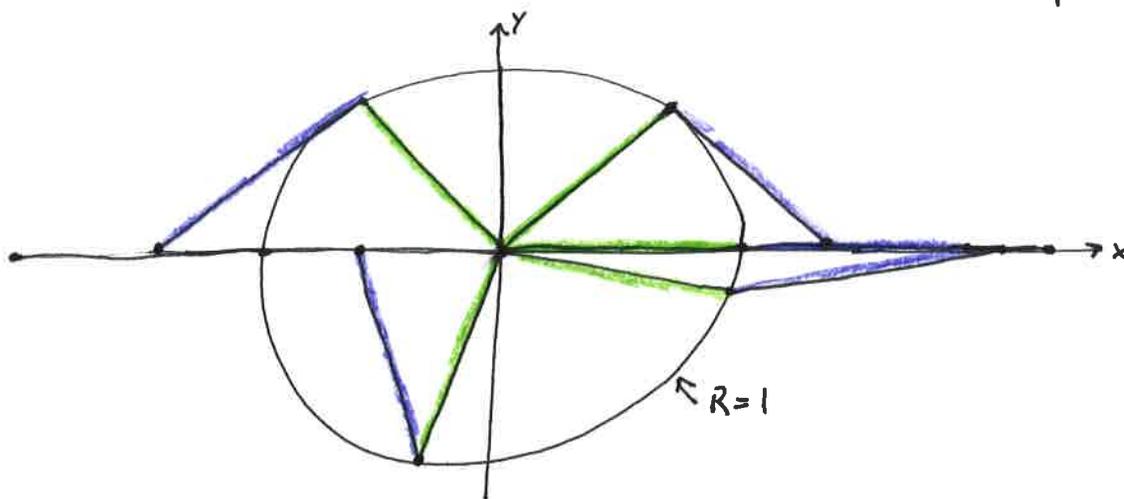
$$J(z) = \underbrace{r e^{i\theta}}_A + \underbrace{\frac{c_1^2}{r} e^{-i\theta}}_B$$

Think of the transform as having two arms or vectors. Arm A traces the unmapped curve (a circle for us). Arm B starts at the end of A, is scaled by c_1^2/r , and is oriented in the conjugate as A's angle.

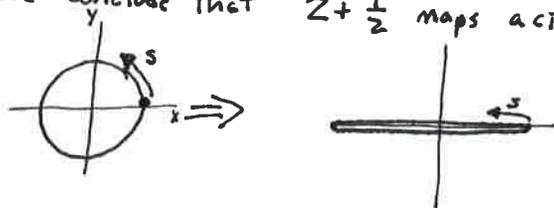


Ex:

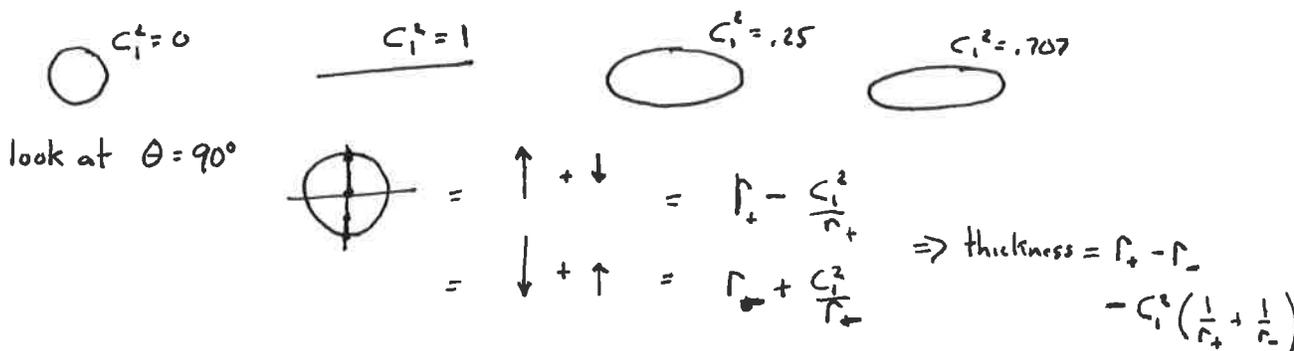
Apply $J(z) = z + \frac{1}{z}$ to a unit circle at $x, y = 0, 0$ $\frac{c_1^2}{r} = 1$



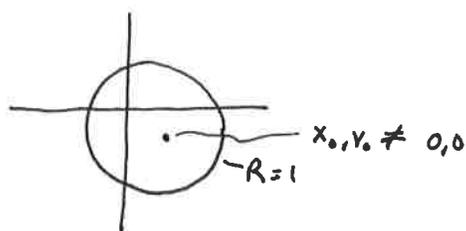
From this, we conclude that $z + \frac{1}{z}$ maps a circle at the origin to an infinitesimally thin airfoil with twice the chord.



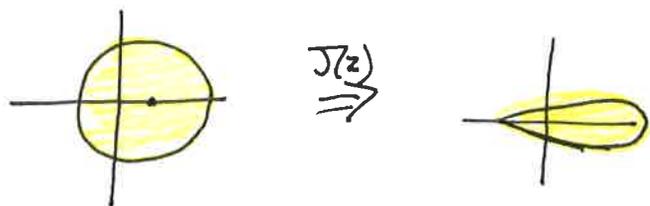
Varying C_1^2 affects the transformed airfoil's thickness (and chord)



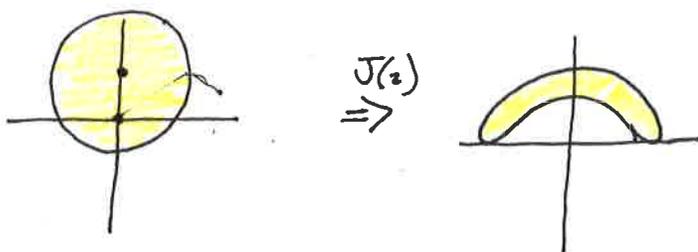
Also, the center of the circle can be offset



- Horizontal offset generates distinctive leading and trailing edges (when $C_1^2 < 1$)



- Vertical offset generates camber



- Combinations generate cambered airfoils

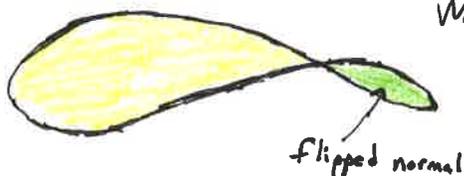


$$x, y = -0.16, 0.23$$

$$C_1 = 0.8$$

Q: Could we call this a J162380?

A: I've never ~~heard~~ seen this terminology!



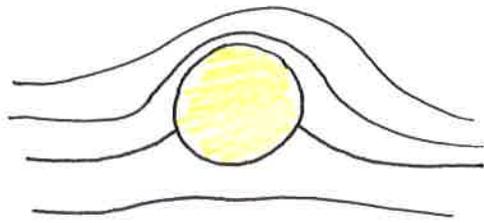
Warning!! Not every combination gives a valid airfoil

$$x, y = -0.32, 0.23$$

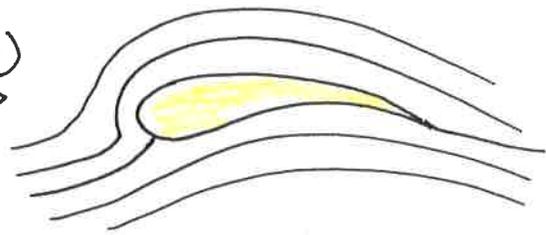
$$C_1 = 0.8$$

Cylinder with Circulation

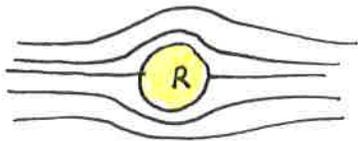
Now, we have the proper motivation to re-study the ~~circle~~ circle-potential flow.



$J(z)$
 \Rightarrow



Remember from fluids,

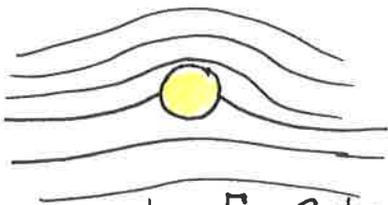


$$\psi = +V \left(r - \frac{R^2}{r} \right) \sin \theta$$

$$= \underbrace{+Vr \sin \theta}_{\text{Uniform flow}} + \underbrace{-V \frac{R^2}{r} \sin \theta}_{\text{Doublet}}$$



Adding a vortex at the center of the circle gives

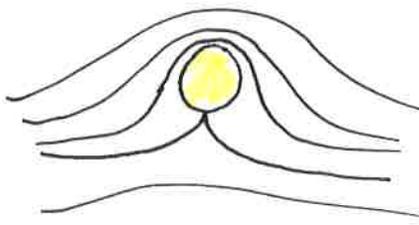


Low Γ , 2 stagnation pts.

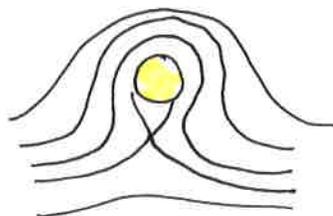
$$\psi = Vr \sin \theta - \frac{VR^2}{r} \sin \theta + \underbrace{\frac{\Gamma}{2\pi r}}_{\text{vortex}} \theta$$



Remember that adding a vortex is acceptable since the domain we want can be isolated from the vortex with a branch cut.



High Γ , 1 stagnation pt
($\Gamma = 4\pi V_0 R$)



Super high Γ , no stagnation pt on body.

So, which Γ do we pick?

Potential Function

- Take velocity potential and stream function and combine into a potential function

$$w = \phi + i\psi$$

Warning!!
W is not $\nabla \times V$ or V_z

For our cylinder (uniform + doublet + vortex),

$$w = -V \left(z + \frac{R^2}{z} \right) - \frac{i\Gamma}{2\pi} \ln \frac{z}{R}$$

$$\text{thus } \phi = \text{Real}(w)$$

$$\psi = \text{Imag}(w)$$

- Write the velocity as a complex #

$$q = u + iv \quad \text{and} \quad \bar{q} = u - iv$$

$$|q| = |u + iv| = |u - iv| = \sqrt{u^2 + v^2}$$

From definition of ϕ and ψ ,

$$u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

Thus

$$\begin{aligned} |q| &= |u + iv| = \left| \frac{\partial \phi}{\partial x} + i(-1)\left(\frac{\partial \psi}{\partial x}\right) \right| \quad \text{or} \\ &= |u - iv| = \left| \frac{\partial \phi}{\partial x} - i(-1)\frac{\partial \psi}{\partial x} \right| = \left| \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} \right| = \left| \frac{\partial}{\partial x}(\phi + i\psi) \right| \\ &= \left| \frac{\partial}{\partial x}(w) \right| = \boxed{\left| \frac{\partial w}{\partial x} \right| = |q|} \end{aligned}$$

Also

$$|q| = \left| \frac{\partial w}{\partial x} \right| = \left| \frac{\partial w}{\partial z} \frac{dz}{dx} \right| \quad \text{but } z = x + iy \Rightarrow \frac{dz}{dx} = 1$$

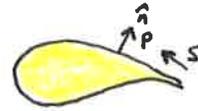
$$\boxed{|q| = \left| \frac{\partial w}{\partial z} \right|}$$

Lift and Moments in a potential flow

Given our mapping $\circ \xrightarrow{J(z)} \omega$, we need the lift, drag and moment generated.

1) One method is to calculate pressures and surface normals and integrate

$$\text{Force} = \oint_{\text{Airfoil}} p \cdot \hat{n} \, dS$$



How can we find \hat{n} ?

2) Apply momentum equation in a CV enclosing the airfoil. (EVA eqn 1.28)

$$\underbrace{\iiint \frac{d\rho v}{dt} dV}_{\text{steady state}} + \iint p(\mathbf{v} \cdot \mathbf{n}) \, dS = \iiint \rho \mathbf{f} \, dV + \iint -p \hat{n} \, dS + \underbrace{\iint \vec{\tau} \cdot \hat{n} \, dS}_{\text{No viscosity}} + \underbrace{B}_{\text{forces acting on fluid airfoil}}$$

Simplify to

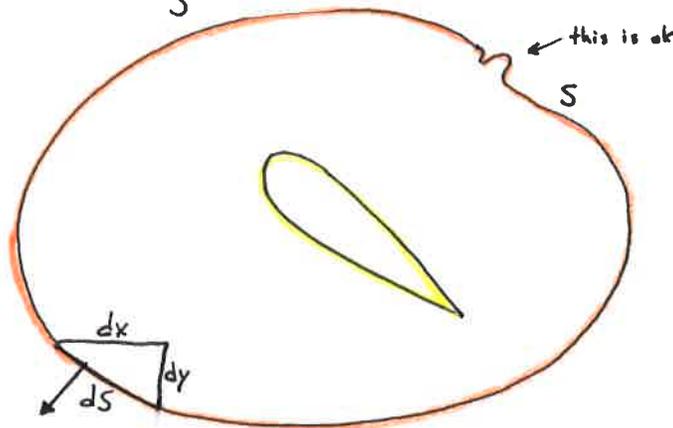
$$\int_S p (u n_x + v n_y + w n_z) \begin{pmatrix} u \\ v \\ w \end{pmatrix} dS = - \int_S p \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} dS + \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{2D}$$

force on airfoil = +B

Rearrange to solve for X, Y, Z

$$X = + \int_S p n_x \, dS + \rho \int_S (u n_x + v n_y) u \, dS$$

$$Y = + \int_S p n_y \, dS + \rho \int_S (u n_x + v n_y) v \, dS$$



$$\hat{n} = \begin{pmatrix} -dy \\ dx \end{pmatrix} \frac{1}{dS}$$

L+M continued

$$X = - \int_S p dy + \rho \int_S (v dx - u dy) u$$

$$Y = \int_S p dx + \rho \int_S (v dx - u dy) v$$

Compute force in complex frame

$$X - iY = - \int_S p (dy + i dx) + \rho \int_S (u - iv)(v dx - u dy)$$

For low speeds (incomp)

$$p = p_0 - \frac{1}{2} \rho (u^2 + v^2)$$

$$X - iY = - \int_S (p_0 - \frac{1}{2} \rho (u^2 + v^2)) (dy + i dx) - \rho \int_S (u - iv)(u dy - v dx)$$

↙ switch signs from above ↘

But, $\int_S p_0 ds = 0$

Thus

$$X - iY = \frac{1}{2} \rho i \int_S \left(\frac{dw}{dz} \right)^2 dz$$

Remember, $|q| = \left| \frac{dw}{dz} \right|$ is the velocity

Moment:

Similar derivation

$$M = - \frac{1}{2} \rho \operatorname{Real} \left(\int_C \left(\frac{dw}{dz} \right)^2 z dz \right)$$

These are the Blasius equations developed by ... well ... Blasius in 1908.