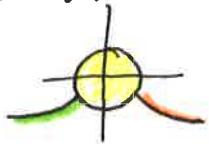


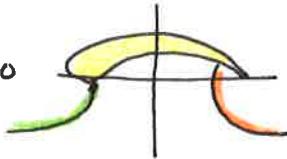
Lesson 7 part 2
Joukowski

How much Γ ??

We have



but $J(z)$ transforms to



To satisfy the Kutta Joukowski condition, the aft stagnation streamline must align to the trailing edge.

Why?

Real fluids have viscosity. If the TE is not a stagnation point, then the sharp TE generates infinite velocity/acceleration.



$$\frac{V^2}{R} = \frac{V}{\frac{V_{\infty}}{\omega}} \text{ undefined.}$$

A real fluid would have a separation point instantly, thus bringin the streamline back to the TE, since ω convects downstream.

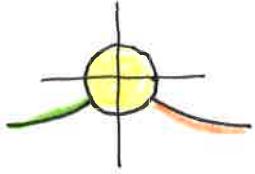


\Rightarrow

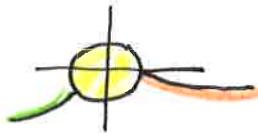


Plus, K-J is what we observe in the wind-tunnel. Almost....

So, to align the TE, which is the location where the unit circle crosses the x-axis, with the stagnation point, we rotate the entire cylinder solution.



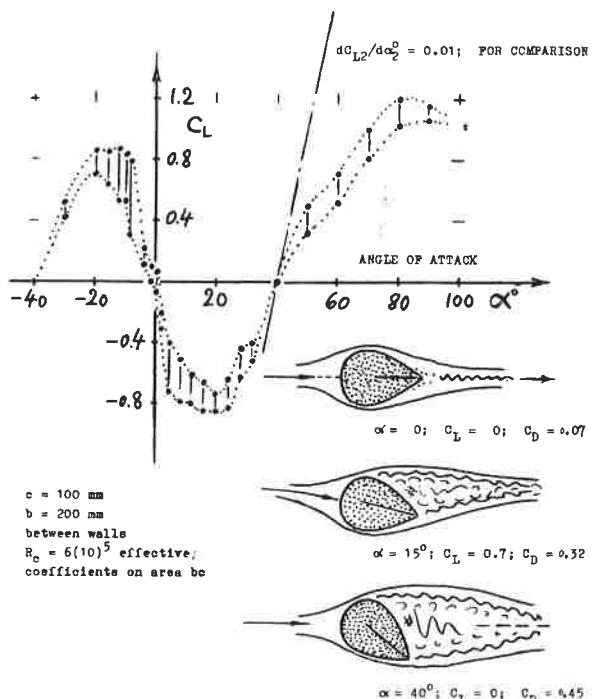
rotate ccw to



How much Γ again

What if we have a viscous case where the effective flow angle does not line up with the geometry?

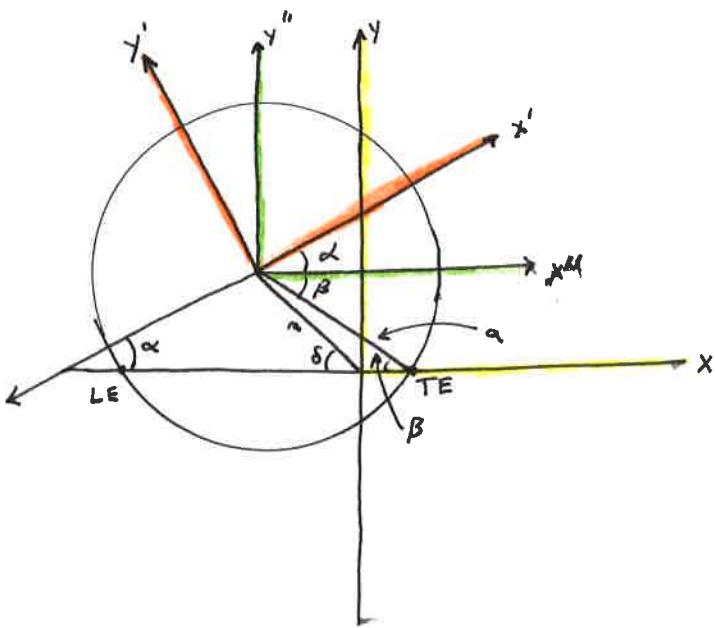
Case: NACA 0070 (70% thick "strut")



C_{L_d} is negative for $|\alpha| < 20^\circ$!!

Figure 16. Lift of the extremely thick foil or strut section 0070, tested (18) to an angle of 90%. The lower and upper points plotted, correspond to the time-dependent fluctuations of the separated flow pattern. R'number above critical.

STRUT SECTION. "Streamline" sections, suitable to be applied in struts or in propeller blades near the hub, have usually higher thickness ratios than conventional foil sections. As an extreme example, the lift coefficient of an 0070 section (18) is presented in figure 16. The lift-curve slope at small angles of attack is strongly negative (!) up to $\alpha \approx 20^\circ$. The flow pattern proves that negative lift is the result of flow separation from the upper side of the section. As a consequence of the well-attached flow along the negatively cambered lower side of the section, suction develops there, thus producing negative lift. As the angle of attack is increased, the lower side eventually produces predominantly positive pressures and a correspondingly positive lift. Also note that the maximum lift coefficient at $\alpha = 90^\circ$, fluctuating between 1.0 and 1.2, corresponds to suction forces developing around the section's nose. - It is shown in (22,b) how the lift function of an airfoil with $t/c = 68\%$, is almost perfectly linear, with a blunt trailing edge (see later). It can also be found in Chapter III of "Fluid-Dynamic Drag," that the



$$z' = r'e^{i\theta'}$$

$$Z'' = Z'e^{i\alpha} \quad \text{since } Z'' = r'e^{i(\beta' + \alpha)} = r'e^{i\beta'}e^{i\alpha}$$

$$Z = Z'' - m e^{i\delta} = Z' e^{\alpha} - m e^{i\delta}$$

How much Γ ? Enough to make the TE a stagnation point (i.e. $V=0$)

$$w = -V \left(z' + \frac{R^2}{z'} \right) - \frac{i \Gamma}{2\pi} \ln \frac{z'}{R}$$

$$\frac{dw}{dz} = -V \left(1 - \frac{R^2}{z'^2} \right) - \frac{i\pi}{2\pi z'}$$

and the velocity magnitude $|g|$ is $|g| = \sqrt{\frac{d\omega}{dz}}$

$$\text{Thus, at the } J_E^E = -V + \frac{VR^2}{Z'^2_{TE}} - \frac{i\Gamma}{2\pi Z'_{TE}}$$

$$Z'_{TE} = a e^{-i(\alpha + \beta)} \quad (Z'_{TE})^2 = a^2 e^{-2i(\alpha + \beta)}$$

$$-V + \frac{VR^2}{a^2} e^{2i(\alpha+\beta)} - \frac{i\Gamma}{2\pi a} e^{-i(\alpha+\beta)} = 0 \quad \text{also } a=R$$

Solve for π

$$\Gamma = \frac{-2\pi R V e^{-i(\alpha+\beta)} + \frac{VR^2}{R^2} Re^{-i(\alpha+\beta)} e^{2i(\alpha+\beta)} 2\pi}{i} = 2 \cdot 2\pi RV \underbrace{\left(e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)} \right)}_{2i}$$

Remember this?

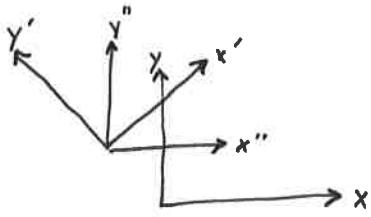
$$F = 4\pi R V \sin(\alpha + \beta)$$

Transform forces and moments to $J(z)$ frame.

$$J(z) = z + \frac{C_1^2}{z}$$

From geometry

$$z' = r'e^{i\theta'}$$



$$z'' = z'e^{i\alpha}$$

$$z = z'' - m\bar{e}^{i\delta} = \underbrace{z'e^{i\alpha}}_{z'} - m\bar{e}^{i\delta} = r'e^{i\theta''} - m\bar{e}^{-i\delta}$$

solve for z'

$$z'e^{i\alpha} = z + m\bar{e}^{i\delta} \Rightarrow z' = ze^{i\alpha} + m\bar{e}^{-i\delta}e^{-i\alpha} = e^{-i\alpha}(z + m\bar{e}^{i\delta})$$

Thus,

$$\frac{dz'}{dz} = e^{-i\alpha}$$

And

Calculate $\frac{dw}{dJ}$ to determine velocity magnitude

$$\frac{dw}{dJ} = \frac{dw}{dz'} \frac{dz'}{dz} \frac{dz}{dJ}$$

↗ we know this from cylinder ↗ Joukowski map
 ↗ offset and rotation map

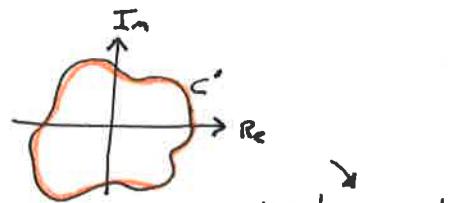
$$\frac{dw}{dJ} = \left(-V + \left(\frac{R^2}{z'^2} \right) - \frac{i\Gamma}{2\pi z'} \right) \left(e^{-i\alpha} \right) \left(1 - \frac{C_1^2}{z^2} \right)^{-1}$$

$$= \left(-V + \frac{R^2}{\bar{e}^{2i\alpha}(z + m\bar{e}^{i\delta})^2} - \frac{i4\pi RV \sin(\alpha + \beta)}{\bar{e}^{i\alpha}(z + m\bar{e}^{i\delta})} \right) \left(e^{-i\alpha} \right) \left(1 - \frac{C_1^2}{z^2} \right)^{-1}$$

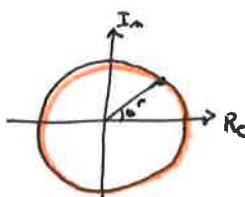
This gets messy....

Cauchy's 2nd Theorem

$$f(z) = \int_{C'} z^n dz$$



$$= \int_C z^n dz$$



• $n \neq -1$

$$= \int_{\theta=0}^{2\pi} (r^n e^{in\theta}) i r e^{i\theta} d\theta$$

$$z = r e^{i\theta}$$

$$dz = i r e^{i\theta} d\theta$$

$$= \int_0^{2\pi} i r^{n+1} e^{i(n+1)\theta} d\theta = i r^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} d\theta = (i r^{n+1}) \left. \frac{e^{i(n+1)\theta}}{i(n+1)} \right|_0^{2\pi}$$

$$= i r^{n+1} \left(\frac{e^{i(n+1)2\pi} - 1}{i(n+1)} \right)$$

$e^{i2\pi \cdot (n+1)}$ is just multiples of ~~2π~~
of revolutions around the Re-Im axis
~~= 1~~ = 1

$$= 0 \text{ unless } n = -1$$

• $n = -1$

$$f(z) = \int_C z^n dz = \int_C \frac{dz}{z} = \int_0^{2\pi} \frac{i r e^{i\theta}}{r e^{i\theta}} d\theta = \int_0^{2\pi} i d\theta = i 2\pi$$

Summary

and a curve that encloses the point where $\bar{z}^1 \rightarrow \infty$ ("singular point")

Given a complex "polynomial" \checkmark $\quad g(z) = a_n z^n + \dots + a_2 z^2 + a_1 z^1 + a_0 z^0 + \dots$

The only non zero term in $f(z) = \int_C g(z) dz$ is the a_1 term!

$$f(z) = \int (a_2 z^2 + a_1 z^1 + a_0 z^0 + \dots) dz = 2\pi i a_1$$

The force equation is

$$X - iY = \frac{1}{2} \rho i \oint \left(\frac{dw}{dz} \right)^2 \frac{dJ}{dz} dz \quad \text{and} \quad M = -\frac{1}{2} \rho \operatorname{Re} \left(\int_c \left(\frac{dw}{dz} \right)^2 J dz \right)$$

From the Cauchy 2nd theorem, we only need to track the $\frac{1}{z}$ terms!!

The Joukowski airfoil involves some algebra and substitution. So, let's focus on the fundamental with the untransformed, non-translated, non-rotated cylinder with a vortex.



$$w = -V \left(z + \frac{R^2}{z} \right) - i \frac{\Gamma}{2\pi} \ln \frac{z}{R}$$

$$\frac{dw}{dz} = -V \left(1 + -\frac{R^2}{z^2} \right) - i \frac{\Gamma}{2\pi z}$$

$$\left(\frac{dw}{dz} \right)^2 = V^2 \underbrace{\left(1 - \frac{R^2}{z^2} \right)^2}_{1 - 2\frac{R^2}{z^2} + \frac{R^4}{z^4}} + \frac{2Vi\Gamma}{2\pi z} + \frac{i^2 \Gamma^2}{(2\pi)^2 z^2}$$

$$\begin{aligned} X - iY &= \frac{1}{2} \rho i \oint \left(\frac{dw}{dz} \right)^2 dz \\ &= \frac{1}{2} \rho i \oint \left(V^2 - 2V^2 \frac{R^2}{z^2} + V^2 \frac{R^4}{z^4} + \frac{2Vi\Gamma}{2\pi z} + \frac{i^2 \Gamma^2}{(2\pi)^2 z^2} \right) dz \end{aligned}$$

From the Cauchy theorem, only the $\frac{1}{z}$ terms are non zero.

$$= \frac{1}{2} \rho i \int \underbrace{\frac{2Vi\Gamma}{2\pi z}} dz + (\checkmark \cdot)^6$$

$$q_1 = \frac{2Vi\Gamma}{2\pi}$$

$$= \frac{1}{2} \rho i (2\pi i) \left(\frac{2Vi\Gamma}{2\pi} \right) = -i\rho V \Gamma$$

$$X - iY = -i\rho V \Gamma$$

$$\boxed{\begin{aligned} X &= 0 \\ Y &= \rho V \Gamma \end{aligned}}$$

The lift is $\rho V \Gamma$
The drag is zero

Returning to Joukowski (we left the algebra elves to work....)

Integrating $X - iY = \frac{1}{2} \rho i \int \left(\frac{dw}{dz} \right)^2 \frac{d\bar{z}}{dz} dz$ ← very messy even when only considering the $\frac{1}{z}$ terms.

Gives

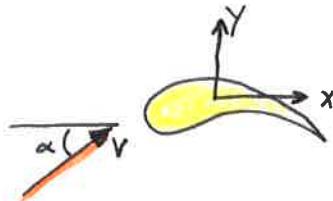
$$X - iY = \frac{1}{2} \rho i (2\pi i) \frac{i V \Gamma e^{i\alpha}}{\pi}$$

$$= -i \rho V \Gamma e^{i\alpha}$$

$$= -i \rho V \Gamma (\cos \alpha + i \sin \alpha)$$

$$X = \rho V \Gamma \sin \alpha$$

$$Y = \rho V \Gamma \cos \alpha$$



But remember that X and Y are in the body frame, not the freestream frame

X and Y are components ($\sin \alpha, \cos \alpha$) of a vector perpendicular to V . This is the lift vector. Zero drag.



$$L = \cancel{\frac{1}{2}} \rho V \Gamma$$

Remember that we found Γ as a function of the geometry necessary for $V=0$ at TE.

$$\Gamma = 4\pi R V \sin(\alpha + \beta)$$

Substitute

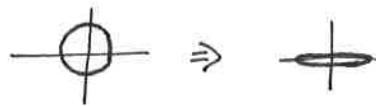
$$L = \rho V 4\pi R V \sin(\alpha + \beta) = \underbrace{\frac{1}{2} \rho V^2}_{\text{q}} \cdot \underbrace{4R}_{\substack{\text{related} \\ \text{to} \\ \text{chord}}} \cdot \underbrace{2\pi}_{\text{constant}} \cdot \underbrace{\sin(\alpha + \beta)}_{\substack{\text{angle} \\ \approx \alpha \text{ for small } \alpha + \beta}}$$

Likewise, the moment at the center of the circle gives:

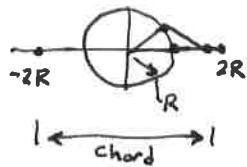
$$M_{cc} = \frac{1}{2} \rho V^2 \cdot 4\pi C_l \cdot \sin 2\alpha$$

Flat plate airfoil

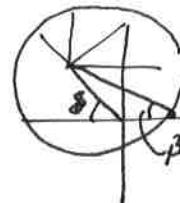
This is the transform with $C_i^2 = R^2$ and no circle offset.



The airfoil has a chord of $4R$ since $J = 2 + \frac{1}{2} = Re^{i\theta} + \frac{R^2}{R^2} e^{-i\theta} = R(e^{i\theta} + e^{-i\theta})$



Als



The circle is not offset
thus $\beta = 0$

Lift

$$L = \frac{1}{2} \rho V^2 \cdot 4R \cdot 2\pi \cdot \sin(\alpha + \beta)$$

$$C_L = \frac{L}{g \bar{c}} = \frac{\frac{1}{2} \rho V^2 \cdot 4R \cdot 2\pi \cdot \sin(\alpha)}{\frac{1}{2} \rho V^2 \cdot 4R} \approx 2\pi \alpha - 2\pi \frac{\alpha^3}{6} + 2\pi \frac{\alpha^5}{120}$$

2 π α is an approximation!

Good to 1% at 14° \leftarrow stall
2% at 20°
10% at 45°

Moment at midchord and $\frac{1}{4}$ chord

$$M_b = \frac{1}{2} \rho V^2 4\pi C_i^2 \sin(2\alpha)$$

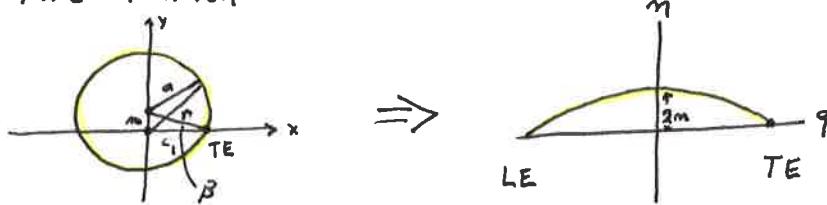
$$C_{m_{\frac{1}{2}}} = \frac{M}{g \bar{c}^2} = \frac{\frac{1}{2} \rho V^2 4\pi C_i^2 \sin(2\alpha)}{\frac{1}{2} \rho V^2 (4C_i)^2} = \frac{\pi}{4} \sin 2\alpha$$

At $\frac{1}{4}$ chord,

$$\begin{aligned} C_{m_{\frac{1}{4}}} &= C_{m_{\frac{1}{2}}} - \frac{C_L}{4} = \frac{\pi}{4} \sin(2\alpha) - \frac{2\pi \sin(\alpha)}{4} \\ &= \frac{\pi}{2} \sin \alpha (\cos \alpha - 1) \end{aligned}$$

≈ 0 for low values of α

Circular Arc Airfoil



$$x^2 + (y - m)^2 = a^2$$

and
 $c_1^2 + m^2 = a^2 \Rightarrow m = 2m \sin^2 \theta$

Define $f \equiv 2m$, thus camber as a percent of chord is $\frac{f}{c} = \frac{2m}{4c_1}$

Lift:

$$\begin{aligned} C_L &= 2\pi \sin \left(\alpha + \frac{\text{atan } \frac{m}{c_1}}{2} \right) \approx 2\pi \left(\alpha + \frac{2f}{c} \right) \\ &= 2\pi \sin \left(\alpha + \text{atan} \left(\frac{4mf}{c^2} \right) \right) \\ &= 2\pi \sin \left(\alpha + \text{atan} \frac{2f}{c} \right) \end{aligned}$$

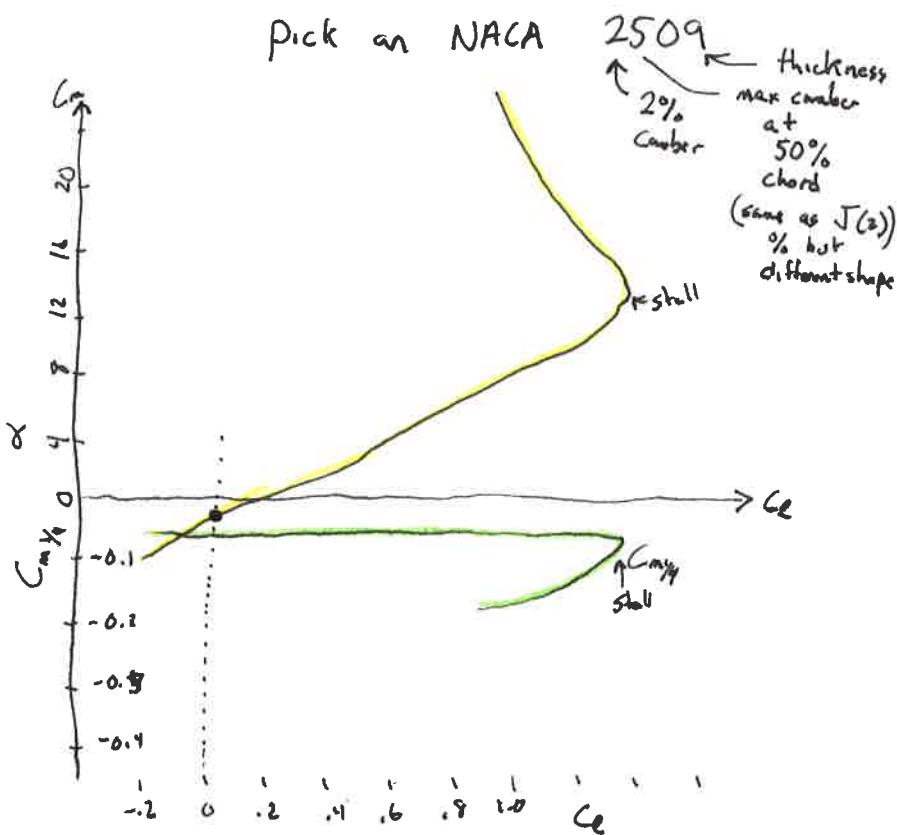
Adding camber increases lift. A cambered airfoil has a zero lift angle of $-\frac{2f}{c}$

Moment:

$$\begin{aligned} C_m y_c &\approx -\frac{\pi f}{c} \quad \text{Adding camber introduces a negative moment.} \\ &= -\frac{\pi}{2} \text{atan} \frac{2f}{c} \end{aligned}$$

Compare to experiments

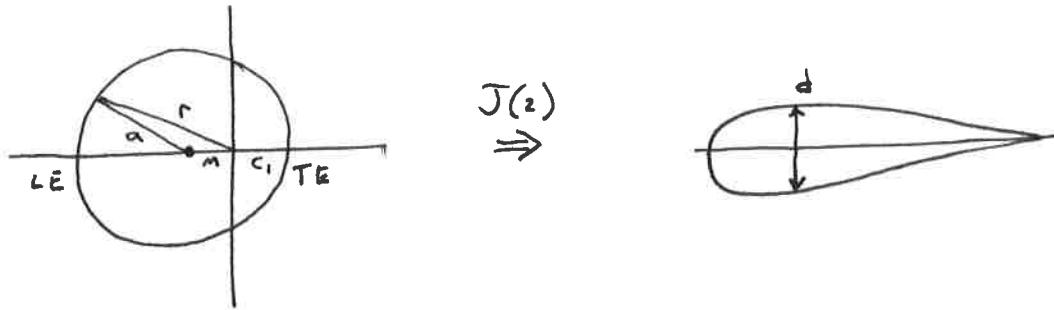
Pick an NACA



$$\alpha_{zL} \approx -\frac{2f}{c} = -0.04 \text{ rad} \approx -2.3^\circ \checkmark$$

$$\begin{aligned} C_m y_c &\approx -\frac{\pi f}{c} = -\pi \cdot 0.02 \\ &\approx -0.063 \checkmark \end{aligned}$$

Symmetrical Airfoil



$$\text{Define } \epsilon = \frac{m}{c_1} \quad \text{then} \quad a = c_1(1 + \epsilon) = c_1 + m \quad \text{and} \quad r = a + m$$

$$= c_1(1 + \epsilon) + m$$

$$= c_1(1 + \epsilon) + c_1\epsilon$$

$$= c_1(1 + 2\epsilon)$$

What is the chord?

Nose: $\zeta = z + \frac{c_1^2}{2} = re^{i\theta} + \frac{c_1^2}{r}e^{-i\theta} =$

$$= -c_1(1 + 2\epsilon) - \frac{c_1^2}{c_1(1 + 2\epsilon)} \approx -2c_1(1 + 2\epsilon^2)$$

Trailing Edge:

$$\zeta = z + \frac{c_1^2}{2} = \dots \approx 2c_1(1 + 2\epsilon^2)$$

$$\text{Chord} \approx 4c_1(1 + 2\epsilon^2)$$

What is the thickness? What is the ϵ term?

$$d = 3\sqrt{3} \text{ m} \quad (\text{from some math}) \Rightarrow \frac{d}{c} = \frac{3\sqrt{3} c_1 \epsilon}{4 c_1 (1 + 2\epsilon^2)} \approx \frac{3\sqrt{3}}{4} \epsilon$$

$$\boxed{\epsilon \approx \frac{4}{3\sqrt{3}} \frac{d}{c}} \approx 77\% \frac{d}{c}$$

Max thickness is near 25% chord

Now for the stunning part....

Sym Airfoil (cont)

$$L = \frac{1}{2} \rho V^2 4c_1 2\pi \sin(\alpha + \beta) \xrightarrow{\text{sym.}}$$

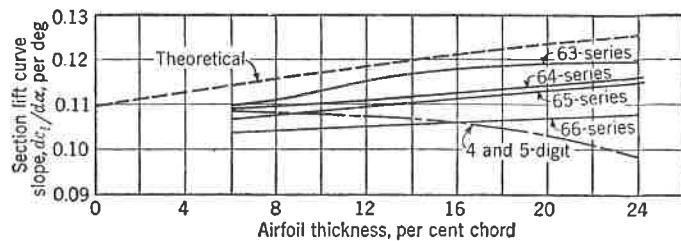
$$C_L = \frac{L}{(\frac{1}{2} \rho V^2 c)} = \frac{\frac{1}{2} \rho V^2 4c_1 (1+\epsilon) 2\pi \sin \alpha}{\frac{1}{2} \rho V^2 4c_1 (1+2\epsilon^2)}$$

$$C_L \approx 2\pi \frac{(1+\epsilon)}{(1+\epsilon^2)} \alpha$$

$$\Rightarrow C_{L\alpha} \approx 2\pi \frac{1+\epsilon}{1+\epsilon^2} \quad \text{where } \epsilon \approx 77\% \frac{d}{c}$$

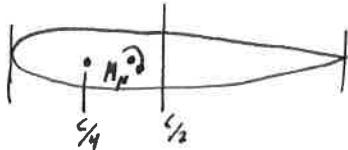
The slope of the lift curve increases with thickness!!

Compare with experiments



Moment:

You might guess that $C_{My_1} = 0$. ~~Consider~~ Consider the following:



The transformation does not preserve lengths from the circle to the airfoil. The moment derived at the circle center is not the half chord.

$$M_p = \frac{1}{2} \rho V^2 4\pi c_1^2 \sin(2\alpha)$$

but

N is $c - (2+\epsilon)c_1$ from the LE

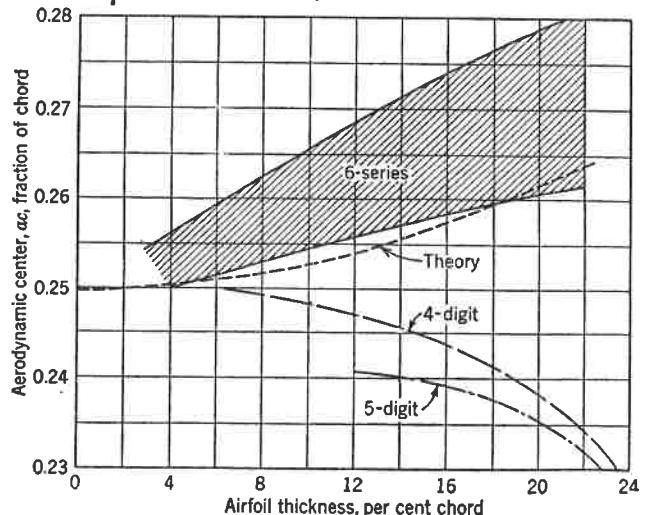
$$M_{ac} = M_p - L(c - (2+\epsilon)c_1 - ac_{true}c)$$

Some math ... $\frac{dM}{dc} = 0$

$$ac_{true} \approx \frac{1}{4} + \frac{\epsilon^2}{2}$$

Adding thickness moves the true aerodynamic center aft.*

Comparison to experiments



* for a Joukowski airfoil