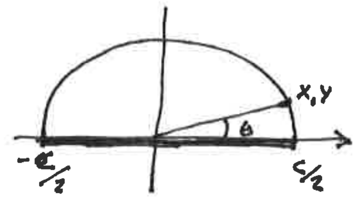
Lesson 10 part 2

Thin Airfoil Theory

Thin Airfoil Theory - Part 2

Cosine spacing - a coordinate transform from x to θ

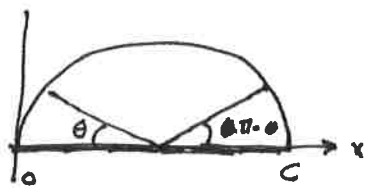


$$X = \frac{c}{2}\cos\theta \quad \Rightarrow dx = -\frac{c}{2}\sin\theta \, d\theta$$

X_{LE} at $\theta = \pi$

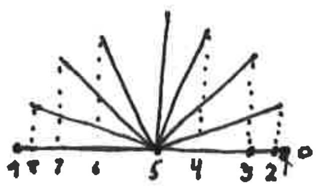
X_{TE} at $\theta = 0$

or



$$X = \frac{c}{2}\left(1 - \cos\theta\right) \Rightarrow dx = \frac{c}{2}\sin\theta \, d\theta$$

This transform is common in aerodynamics. This is also an excellent way to place points on a line segment when you need more resolution at the ends.



8 steps $\Rightarrow \Delta\theta = \pi/8 = 22.5°$

9 points at

0%, 3.8%, 15%, 30%, 50%, 70%, 85%, 97%, 100%

There are even an entire class of polynomials that exploit this transform called the Chebyshev polynomials

$$T_n(\cos\theta) = \cos(n\theta)$$

These are orthogonal and satisfy the following ODE

$$(1-x^2)y'' - xy' + n^2 y = 0$$

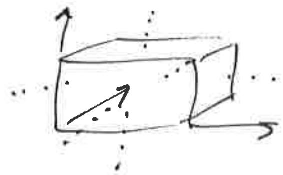
In fact, it has been suggested that if you are not working with harmonic data, the Chebyshev polynomial is ideal (cf. Boyd, Spectral Methods...)

They also have a recurrence equation

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$$

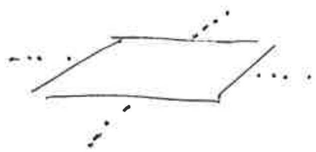
Source and Vortex Sheets in 2D and their perturbation velocities

$$\sigma = \nabla \cdot V \qquad \omega = \nabla \times V \quad \xRightarrow{\text{reverse}} \quad V_\sigma(r) = \frac{1}{4\pi}\iiint \sigma(r') \frac{r-r'}{|r-r'|^3}\,dx'dy'dz'$$



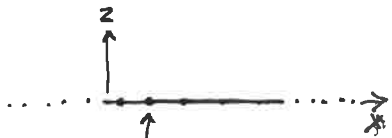
$$V_\omega(r) = \frac{1}{4\pi}\iiint \omega(r') \times \frac{r-r'}{|r-r'|^3}\,dx'dy'dz'$$

$\Downarrow$ sheets



$$V_\sigma(r) = \frac{1}{4\pi}\iint \lambda \frac{r-r'}{|r-r'|^3}\,ds\,dl \qquad \ldots$$

$\Downarrow$ 2D sheets



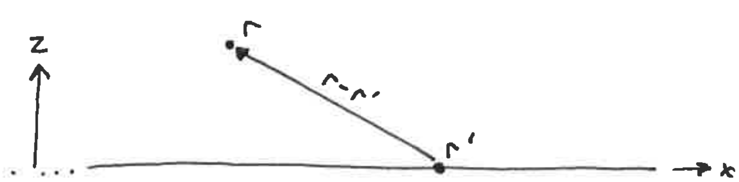
- constant in y direction
- All ω aligned with $\hat{y}$

$$V_\sigma(r) = \frac{1}{2\pi}\iint \sigma(r') \frac{r-r'}{|r-r|^2}\,dx'dz'$$

$$V_\omega(r) = \frac{1}{2\pi}\iint \omega(r') \frac{\hat{y} \times (r-r')}{|r-r'|^2}\,dx'dz'$$

Notice $|r-r'|^2$ not $|r-r'|^3$

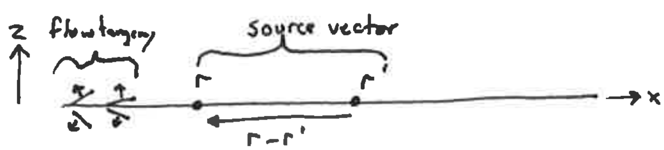
and $\frac{1}{2\pi}$ vs $\frac{1}{4\pi}$



$$r-r' = (x-x')\hat{x} + (z-z')\hat{z}$$

$$|r-r'| = \sqrt{(x-x')^2 + (z-z')^2}$$

Now we see the visual definition of "thin airfoils". All source and vortex properties are determined at and along the x axis. (i.e. $z-z'=0$). Even flow tangency is applied at $z=0$



$$r-r' = (x-x')\hat{x}$$

$$|r-r'| = \sqrt{(x-x')^2} = |x-x'|$$

$$V_\sigma(x) = \frac{1}{2\pi}\int \sigma(x') \frac{(x-x')\hat{x}}{(x-x')^2}\,dx' = \frac{1}{2\pi}\int \sigma(x') \frac{dx'}{x-x'}\,\hat{x}$$

$$V_\omega(x) = \frac{1}{2\pi}\int \omega(x') \frac{\overbrace{\hat{y} \times (x-x')\hat{x}}^{\hat{y}\times\hat{x} = -\hat{z}}}{(x-x')^2}\,dx' = \frac{1}{2\pi}\int \omega(x') \frac{-dx'}{x-x'}\,\hat{z}$$

Perturbation Velocities

Source

$$V_\sigma = \frac{1}{2\pi}\int \sigma(x')\frac{dx'}{x-x'}\,\hat{x} \implies \boxed{u(x) = \frac{1}{2\pi}\int \sigma(x')\frac{dx'}{x-x'}}$$

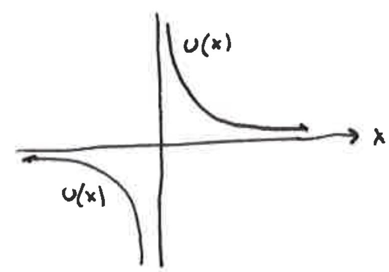
So, along the x axis (and not considering the jump), a source creates a perterbation in the u velocity. Important!

Example:

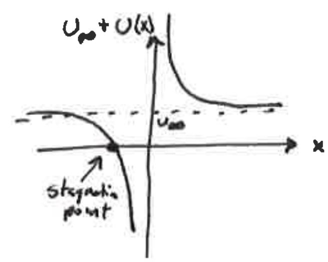
To illustrate this, pick a point source ($\sigma(x') = \delta(x') =$ ) magnitude such that $\int\delta = 1$

$$u(x) = \frac{1}{2\pi}\int \delta(x')\frac{dx'}{x-x'}$$ only has value when $x' = 0$

$$= \frac{1}{2\pi}\int \delta\, dx'\left(\frac{1}{x}\right) = \frac{1}{2\pi x}$$



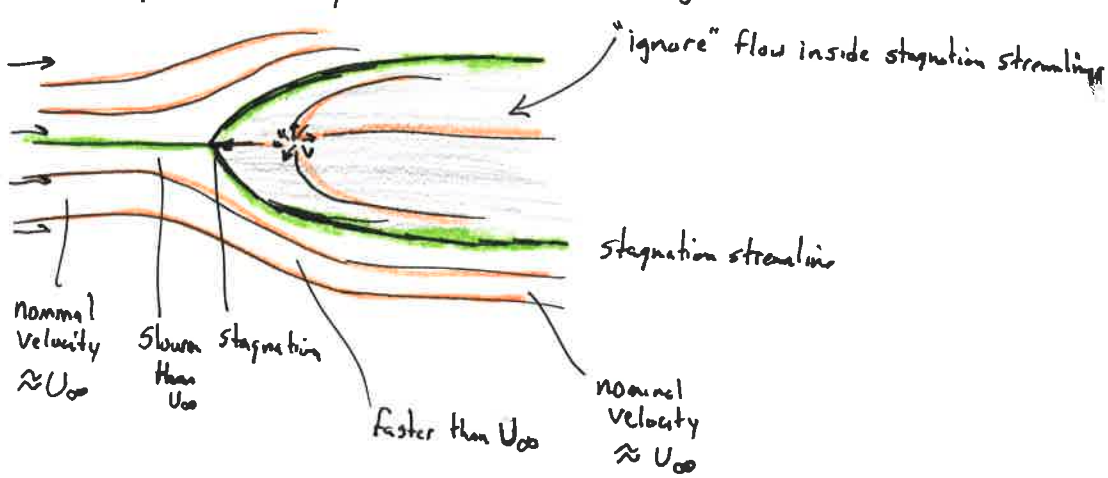
Add a free stream U_∞ $\rightrightarrows$



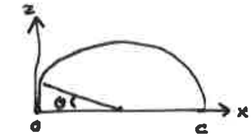
As we approach the point source, the flow velocity decreases to a stagnation point. Aft of the source, the flow velocity decreases from a higher velocity back to freestream.

We have here a proto-symmetrical airfoil.

If we plot in 2D, the stream function ψ gives



Back to our thin airfoil with a cosine transform



$x = \frac{c}{2}(1-\cos\theta)$

$dx = +\frac{c}{2}\sin\theta\, d\theta$

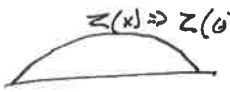
$$U(\theta) = \frac{1}{2\pi}\int_0^c \lambda(x')\frac{dx'}{x-x'} = \frac{1}{2\pi}\int_0^{\pi}\lambda(\theta')\frac{\frac{c}{2}\sin\theta\, d\theta}{\frac{c}{2}(1-\cos\theta)-\frac{c}{2}(1-\cos\theta')}$$

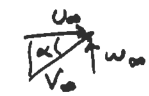
$$= \frac{1}{2\pi}\int_0^{\pi}\lambda(\theta')\frac{\sin\theta' d\theta'}{\cos\theta'-\cos\theta}$$

$$W(\theta) = \frac{1}{2\pi}\int_0^c -\gamma(x')\frac{dx'}{x-x'} = \frac{1}{2\pi}\int_0^{\pi} -\gamma(\theta')\frac{\sin\theta' d\theta'}{\cos\theta'-\cos\theta}$$

Substitute into $\frac{d}{dx}(U_\infty Z) = W_\infty + W$ with $W_\infty = V_\infty \sin(\alpha)$

$U_\infty = V_\infty \cos(\alpha)$





$$\frac{d}{dx}\left(V_\infty \overset{\approx 1}{\cancel{\cos\alpha}}\, Z\right) = V_\infty \overset{\approx \alpha}{\cancel{\sin\alpha}} + \frac{1}{2\pi}\int_0^{\pi} -\gamma(\theta')\frac{\sin\theta' d\theta'}{\cos\theta'-\cos\theta}$$

$$\boxed{\frac{dZ}{dx} = \alpha + \frac{1}{2\pi}\int_0^{\pi} -\frac{\gamma(\theta')}{V_\infty}\frac{\sin\theta' d\theta'}{\cos\theta'-\cos\theta}}$$

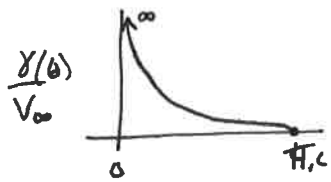
Apply kutta condition

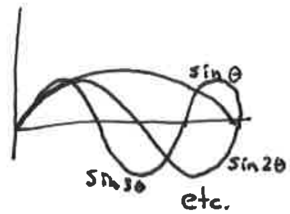
$$\gamma\Big|_{\pi} = 0$$

What is $\gamma(\theta)$?

$$\frac{1}{2\pi}\int_0^{\pi} \frac{\gamma(\theta')}{V_\infty}\,\frac{\sin\theta'\,d\theta'}{\cos\theta' - \cos\theta} = A_0 - \sum_{n=1}^{\infty} A_n \cos n\theta$$

$$\frac{\gamma(\theta)}{V_\infty} = 2A_0\,\frac{1+\cos\theta}{\sin\theta} + 2\sum_{n=1}^{\infty} A_n \sin n\theta$$

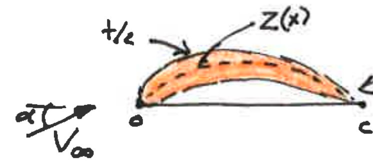




What is $\lambda(x)$?

$$\lambda(x) = V_\infty \frac{dt}{dx}$$

Thin airfoil



Is it required that $Z(\pi)=0$? No, but we pull that offset into α.

$$\boxed{U_\infty \frac{dt}{dx} = \lambda}$$

and

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta')}{V_\infty} \frac{\sin\theta'\, d\theta'}{\cos\theta' - \cos\theta} = \alpha - \frac{dZ(\theta)}{dx} \quad \text{with} \quad \gamma(\pi) = 0$$

Solution:

Represent $\alpha - \frac{dZ(\theta)}{dx}$ with a Fourier series of cos.

Why cos? Well, we want $Z(0) = Z(\pi) = 0$, so a sine series in Z is a cosine series in $\frac{dz}{dx}$.

$$\alpha - \frac{dZ(\theta)}{dx} = A_0 - \sum_{n=1}^{\infty} A_n \cos n\theta$$

From Fourier theory

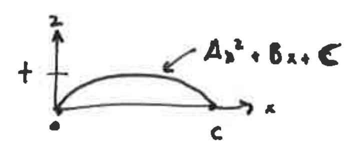
$$A_0 = \alpha - \frac{1}{\pi}\int_0^\pi \frac{dZ}{dx}\, d\theta \quad \text{and} \quad A_n = \frac{2}{\pi}\int_0^\pi \frac{dZ}{dx} \cos n\theta\, d\theta$$

If you can write $\frac{dz}{dx}$ as a Fourier series, Z is also a Fourier series.

Remember that

$$\int_0^\pi \cos n\theta \cos m\theta\, d\theta = \begin{cases} \pi & n=m=0 \\ \pi/2 & n=m\neq 0 \\ 0 & n\neq m \end{cases}$$

Ex: $Z(x) = 4\frac{t}{c}x - 4\frac{t}{c^2}x^2$ for a parabolic camberline.



$$\frac{dZ(x)}{dx} = 4\frac{t}{c} - 8\frac{t}{c^2}x$$

convert to cosine transform/spacing $x = \frac{c}{2}(1-\cos\theta)$

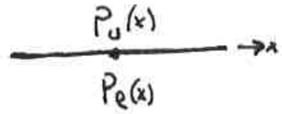
$$\frac{dZ(\theta)}{dx} = 4\frac{t}{c} - 8\frac{t}{c^2}\frac{c}{2}(1-\cos\theta) = 4\frac{t}{c}\cos\theta$$

Match terms

$$A_0 = \alpha - \frac{1}{\pi}\cancelto{0}{\int_0^\pi \cdots\, d\theta} \qquad A_n = \frac{2}{\pi}\int_0^\pi 4\frac{t}{c}\cos\theta \cos n\theta\, d\theta$$

$$\boxed{A_0 = \alpha, \quad A_1 = 4\frac{t}{c}}, \quad A_2 = 0, \quad A_3 = 0, \quad \ldots$$

Forces and Moments



$$\Delta C_p = \frac{P_l - P_u}{q} \qquad \text{from incompressible B'} \qquad \Delta C_p = \frac{|V_u|^2 - |V_\ell|^2}{V_\infty^2}$$

From the velocity jump from the vortex sheet,

$$\Delta C_p \approx 2\frac{\gamma}{V_\infty}$$

Lift:

$$C_\ell = \int_0^c \Delta C_p \, dx \approx \int_0^c 2\frac{\gamma}{V_\infty} dx = \underbrace{\int_0^\pi \frac{\gamma}{V_\infty} \sin\theta \, d\theta}_{\text{cosine transform.}}$$

$$= \int_0^\pi 2\left(A_0 \frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right) \sin\theta \, d\theta$$

with $\int_0^\pi \cancelto{0}{\sin\cos}\, d\theta$ and $\int_0^\pi \sin n\theta \sin m\theta \, d\theta \neq 0$ only when $n = m$

$= \pi/2$ when $n=m$

$$C_\ell = 2\pi A_0 + \pi A_1$$

$$= 2\pi\alpha + 2\int_0^\pi \frac{dZ}{dx}(\cos\theta - 1)\, d\theta$$

$$= 2\pi\alpha + C_{\ell_0}$$ where C_{ℓ_0} depends on the camberline

$$\boxed{\frac{dC_\ell}{d\alpha} = 2\pi}$$