

Lesson 10 part 3

Thin Airfoil Theory

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Joukowski Plotting

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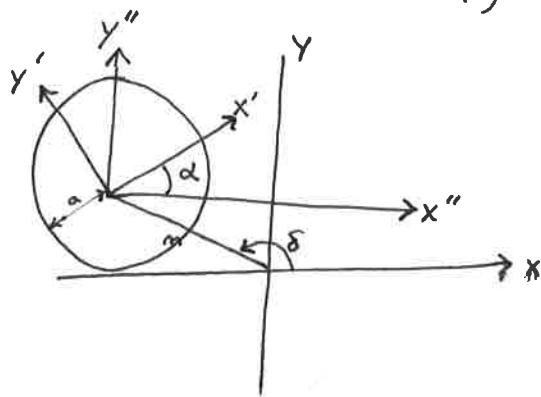
Compressible  $\sigma$

# Joukowski plotting

Q: How can I quickly plot a Joukowski airfoil?

A: Look at the reference frames.

- 1) Our ~~real~~ visible ref frame  $Z = x + iy$
- 2) Translated frame  $Z'' = x'' + iy''$
- 3) Rotated frame (optimal for plotting!!)  $Z' = x' + iy'$



$$\begin{aligned} \text{From } z \text{ to } z'' & \quad Z'' = z - me^{i\delta} \\ \text{From } z'' \text{ to } z' & \quad \text{note the } \underline{\underline{\text{minus}}} \\ Z' &= z'' e^{-i\alpha} \end{aligned}$$

$$\text{Rearrange to } Z = z'' + me^{i\delta} = Z' e^{i\alpha} + me^{i\delta}$$

In the  $z'$  frame, we have a unit circle:  $z' = a e^{i\theta}$

Thus

$$Z = a e^{i\theta} e^{i\alpha} + me^{i\delta} = \underbrace{a e^{i(\theta+\alpha)}}_{\text{Real}} + m e^{i\delta}$$

$$\text{Real} = a \cdot \cos(\theta + \alpha) + m \cos \delta$$

$$\text{Imag} = a \cdot \sin(\theta + \alpha) + m \sin \delta$$

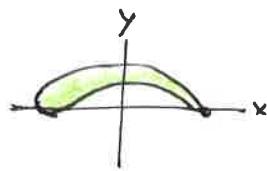
$J(z)$  Transform:  $J(z) = z + \frac{C_1^2}{z}$

$$\begin{aligned} &= x + iy + \frac{C_1^2}{x + iy} = x + iy + \frac{C_1^2(x - iy)}{(x + iy)(x - iy)} = x + iy + \frac{C_1^2(x - iy)}{x^2 + y^2} \\ &= \left( x + \frac{C_1^2 x}{x^2 + y^2} \right) + i \left( y - \frac{C_1^2 y}{x^2 + y^2} \right) \end{aligned}$$

Demo:

Plotting the TE and LE pts.

We know that at the TE,  $y=0$ .

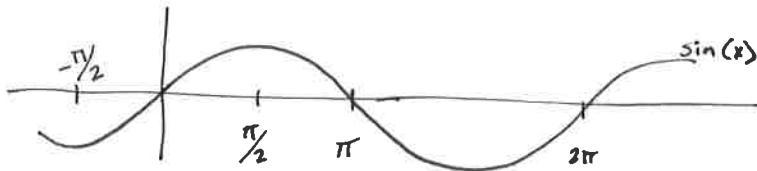


Thus, the imaginary part of  $z$  must be zero

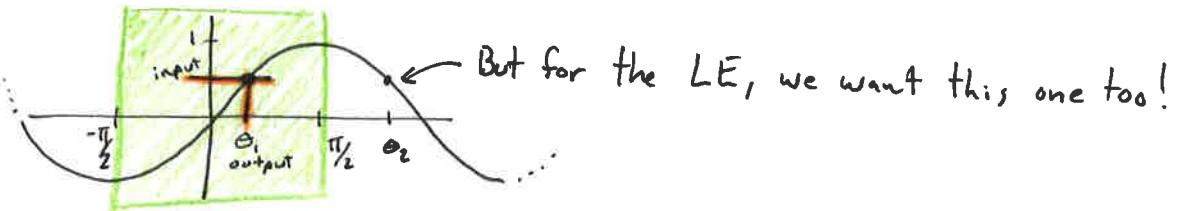
$$a \cdot \sin(\theta) + m \sin(\delta) = 0 \Rightarrow \theta = -\arcsin\left(\frac{m}{a} \sin \delta\right)$$

What about the LE?

What is the behavior of  $\arcsin$ ? Inverse of  $\sin$ !



Numerically, computing  $\arcsin$  will find the principle value which is between  $-\frac{\pi}{2}$  and  $\frac{\pi}{2}$ .



Easy calculation for the LE value of  $\theta$ , since mirrored about  $\frac{\pi}{2}$

$$\theta_2 = \frac{\pi}{2} + \frac{\pi}{2} - \theta_1 = \pi - \theta_1$$

# Extended Thin Airfoil Theory

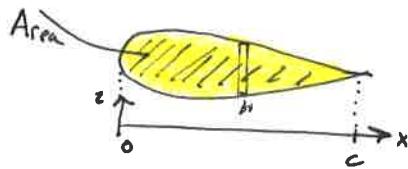
Now, consider the thickness.  $\lambda(x) = \frac{d}{dx}((U_\infty + U) + + \gamma Z) \approx V_\infty \frac{dt}{dx}$

Write thickness as Fourier sine series. Why sine? Zero at endpoints  $(0, c)$

$$t(\theta) = C \sum_{n=1}^{\infty} B_n \sin n\theta$$

$$B_n = \frac{2}{\pi} \int_0^\pi \frac{t}{C} \sin n\theta d\theta$$

Airfoil Area



$$\text{Area} = \int_0^c t dx = \frac{c}{2} \int_0^\pi t \sin \theta d\theta$$

orthogonal, so only  $B_1$  term contributes,

$$= \frac{\pi}{4} C^2 B_1$$

$$\Rightarrow B_1 = \frac{4A}{\pi C^2}$$

Source Sheet

$$\lambda(\theta) = V_\infty \frac{dt}{dx} = V_\infty \frac{dt}{d\theta} \frac{d\theta}{dx} = \dots$$

$$x = \frac{c}{2}(1 - \cos \theta) \Rightarrow \frac{dx}{d\theta} = \frac{c}{2} \sin \theta$$

$$t = C \sum B_n \sin n\theta \Rightarrow \frac{dt}{d\theta} = C \sum n B_n \cos n\theta$$

$$\lambda(\theta) = V_\infty \cdot \frac{2}{c \sin \theta} \cdot C \sum n B_n \cos n\theta = V_\infty \frac{2}{\sin \theta} \sum n B_n \cos n\theta$$

## Velocity Perturbation

$$U(x) = \frac{1}{2\pi} \int_0^c \lambda(x') \frac{dx'}{x-x'}$$

$$= \frac{1}{2\pi} \int_0^\pi V_\infty \frac{2}{\sin \theta} \left[ \sum_{n=1}^{\infty} n B_n \cos n\theta' \right] \cdot \frac{\frac{c}{2} \sin \theta' d\theta'}{\frac{c}{2}(1-\cos \theta') - \frac{c}{2}(1-\cos \theta)}$$

$$U(\theta) = V_\infty \sum_{n=1}^{\infty} n B_n \frac{\sin n\theta}{\sin \theta}$$

Glaert Integral

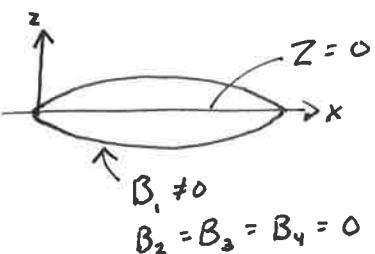
$$\int_0^\pi \frac{\cos n\theta' d\theta'}{\cos \theta' - \cos \theta} = \pi \frac{\sin n\theta}{\sin \theta}$$

$$x' = \frac{c}{2}(1 - \cos \theta')$$

$$dx' = \frac{c}{2} \sin \theta' d\theta'$$

$$\frac{\frac{c}{2} \sin \theta' d\theta'}{\frac{c}{2}(1-\cos \theta') - \frac{c}{2}(1-\cos \theta)}$$

Flat elliptical thickness airfoil



$$\frac{d}{dx} ((U_\infty + U) + \gamma Z) = \lambda$$

$$\frac{d}{dx} ((U_\infty + U) \vec{Z} + \frac{1}{4} \gamma t) = \omega_\infty + \omega$$

thus

$$\frac{1}{4} \frac{d}{dx} (\gamma t) = \omega_\infty + \omega$$

with

$$x = \frac{c}{2}(1 - \cos \theta) \quad (\text{cosine transform})$$

and

$$t = c B_1 \sin \theta \quad (\text{just 1 term})$$

$$\text{and } \frac{d}{dx} = \frac{d}{d\theta} \frac{d\theta}{dx}$$

Plug into  $\frac{1}{4} \frac{d}{dx} (\gamma t) = \omega_\infty + \omega$

$$\frac{1}{4} \frac{2}{c \sin \theta} \frac{d}{d\theta} \left( \frac{\gamma(\theta)}{c B_1 \sin \theta} \right) = V_\infty \sin \theta \vec{d} + \frac{1}{2\pi} \int_0^\pi -\gamma(\theta) \frac{\sin \theta' d\theta'}{\cos \theta' - \cos \theta}$$

differential eqn.                                  integral eqn.

Integro-Differential Eqn. for  $\gamma(\theta)$

Solution!!

$$\frac{\gamma}{V_\infty} = \underbrace{\frac{2\alpha}{1-B_1}}_{\text{Same as flat airfoil mult' by }} \frac{1+\cos \theta}{\sin \theta}$$

Same as flat airfoil mult' by  $\frac{1}{1-B_1}$

## Forces and Moments

We already did this with  $\frac{Y}{V_\infty} = 2\alpha \frac{1+\cos\theta}{\sin\theta}$  !



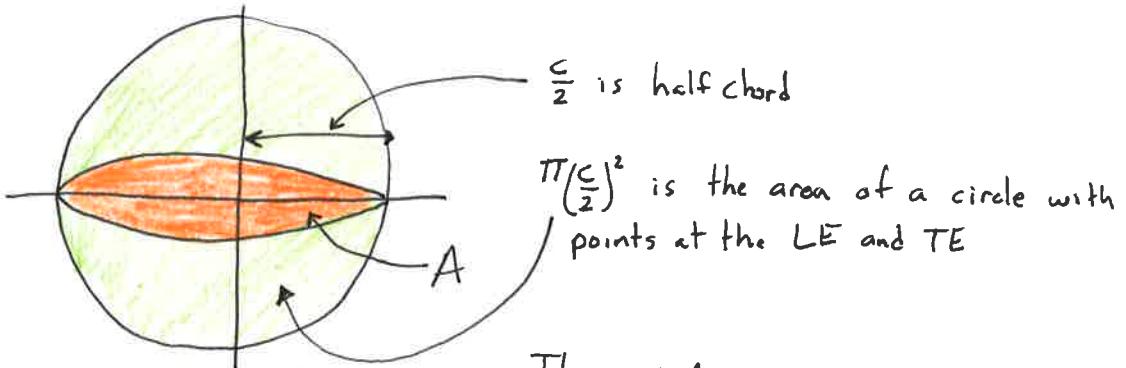
$$\Delta C_p = \frac{P_e - P_0}{\frac{\gamma}{g}} \approx \frac{2\gamma}{V_\infty}$$

Lift

$$\begin{aligned} C_L &= \int_0^C \Delta C_p dx \approx \int_0^C 2 \frac{\gamma}{V_\infty} dx = \int_0^\pi \frac{\gamma}{V_\infty} \sin\theta d\theta \\ &= \int_0^\pi \frac{2\alpha}{1-B_1} \frac{1+\cos\theta}{\sin\theta} \sin\theta d\theta = \int_0^\pi \frac{2\alpha}{1-B_1} (1+\cos\theta) d\theta \\ &= \frac{2\alpha}{1-B_1} \int_0^\pi = \frac{2\pi\alpha}{1-B_1} = (2\pi) \left( \frac{1}{1-4A} \frac{1}{\pi c^2} \right) \alpha \end{aligned}$$

Geometric interpretation of  $\frac{1}{1-4A/\pi c^2}$

$$C_{L\alpha} = \frac{2\pi}{1 - \frac{4A}{\pi c^2}}$$



Thus  $\frac{4A}{\pi c^2}$  is the area ratio of the airfoil and circle.

Does  $C_{L\alpha}$  approach  $\infty$  as the airfoil thickens to a circle? No. This result is only good for thin airfoils.

$$\frac{1}{1-AR} = \text{inverse of } 1 - \text{area ratio}$$

No surprise that the ref Area is a circle since we assumed only  $B_1 \neq 0$  for an elliptical thickness.

## Moment

$$\begin{aligned}
 C_{m\frac{c}{4}} &= \frac{M}{\rho c^2} = \frac{1}{\rho c^2} \int_0^c -\Delta C_p \left( x - \frac{c}{4} \right) dx = \frac{1}{\rho c^2} \int_0^c -\frac{2\gamma}{V_\infty} \left( x - \frac{c}{4} \right) dx \\
 &= \frac{1}{\rho c^2} \int_0^{\pi} -\frac{\gamma}{V_\infty} \left( \frac{c}{2}(1-\cos\theta) - \frac{c}{4} \right) \sin\theta d\theta \\
 &= \frac{1}{\rho c^2} \int_0^{\pi} -\left( \frac{2\alpha}{1-B_1} \right) \left( \frac{1+\cos\theta}{\sin\theta} \right) \left( \frac{c}{2}(1-\cos\theta) - \frac{c}{4} \right) \sin\theta d\theta \\
 &= \frac{1}{\rho c^2} \int_0^{\pi} \left( -\frac{2\alpha}{1-B_1} \right) (1+\cos\theta)(c) \left( \frac{1}{4} - \frac{\cos\theta}{2} \right) d\theta \\
 &= 0 \quad !!!
 \end{aligned}$$

Where is the a.c. shift?

What is the mathematical reason that the integral is zero?

Compare with the cambered thin airfoil

$$\begin{aligned}
 C_{m\frac{c}{4}} &= \frac{1}{\rho c^2} \int_0^{\pi} \left( 2A_0 \frac{1+\cos\theta}{\sin\theta} + 2A_1 \sin\theta + 2A_2 \sin 2\theta + \dots \right) \left( \frac{c}{2}(1-\cos\theta) - \frac{c}{4} \right) \sin\theta d\theta \\
 &= -\frac{\pi c}{4} (A_1 - A_2) \quad \text{where } A_0 \text{ has an } \alpha \text{ term}
 \end{aligned}$$

$$\frac{dC_{m\frac{c}{4}}}{d\alpha} = 0$$

## Compressible flow

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \quad \leftarrow \text{Cons' of mass}$$

$$\nabla \cdot (\rho \mathbf{V}) = \text{"chain rule"} = \rho \nabla \cdot \mathbf{V} + \nabla \rho \cdot \mathbf{V} = 0$$

We previously defined  $\sigma = \nabla \cdot \mathbf{V}$

Thus,  $\rho \sigma + \nabla \rho \cdot \mathbf{V} = 0$

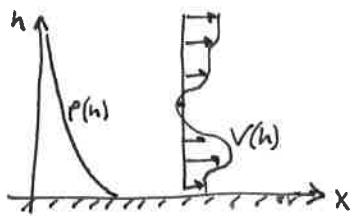
$$\sigma = - \frac{\nabla \rho}{\rho} \cdot \mathbf{V}$$

Ex:

Q: Is a flow compressible?

A: Is  $\nabla \rho$  aligned with  $\mathbf{V}$ ? Is  $\nabla \rho \cdot \mathbf{V} \neq 0$

Ex: Atmosphere + wind



$\nabla \rho$  is in  $-h$  direction  
 $\mathbf{V}$  is in  $x$  direction

$$\sigma = \frac{\nabla \rho \cdot \mathbf{V}}{\rho} = 0$$



From isentropic flow

$$\frac{P_2}{P_1} = \left( \frac{P_2}{P_1} \right)^{\gamma} = \left( \frac{h_2}{h_1} \right)^{\frac{\gamma}{\gamma-1}} \quad \text{or} \quad \frac{dp}{p} = \gamma \frac{dh}{h} = \frac{\gamma}{\gamma-1} \frac{dh}{h}$$

$$\sigma = - \frac{\nabla \rho}{\rho} \cdot \mathbf{V} = - \frac{1}{\gamma-1} \frac{\nabla h}{h} \cdot \mathbf{V}$$

$$\text{with } h = h_0 - \frac{1}{2} V^2 \quad \text{and} \quad a = \sqrt{(\gamma-1)h}$$

$$\sigma = - \frac{1}{(\gamma-1)h} \nabla \left( h_0 - \frac{1}{2} V^2 \right) = \frac{1}{2} \frac{\nabla (V^2)}{a^2} = V^2 \frac{\nabla V}{a^2} \cdot \hat{s}$$

$$\sigma = M^2 \frac{\partial V}{\partial s}$$

$\hat{s}$  along streamline

"All flows are compressible, but high Mach # flows are more compressible"