

Lesson 13

Far-field Analysis

Decomposing a flow field into source, vorticity, and freestream components gives

$$V = V_{\sigma} + V_{\omega} + V_b \quad \text{where } \sigma = \nabla \cdot V$$

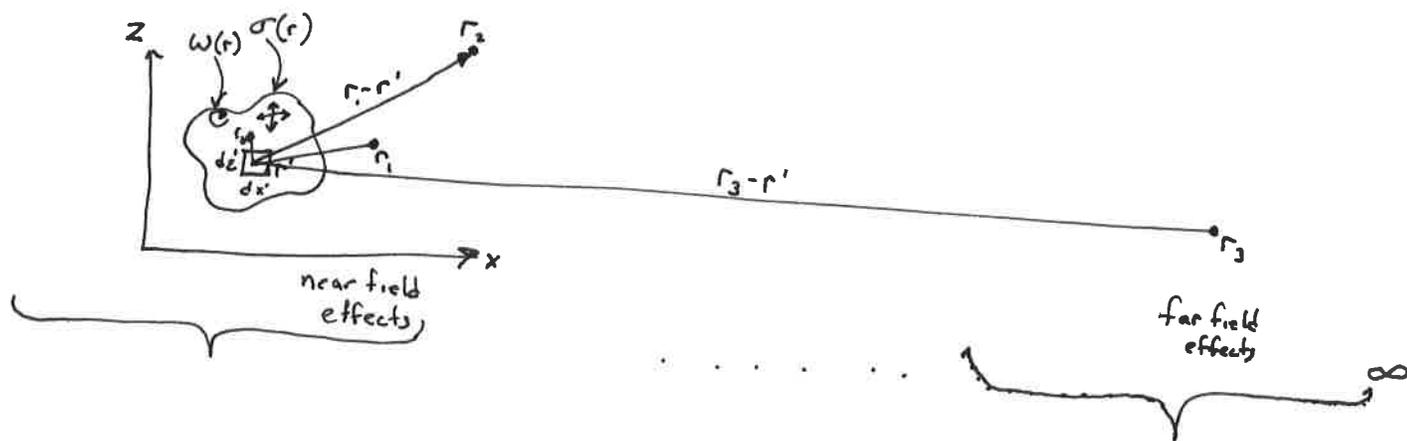
$$\omega = \nabla \times V$$

and

$$V_{\sigma} = \frac{1}{4\pi} \iiint \sigma(r') \frac{r-r'}{|r-r'|^3} dx' dy' dz'$$

$$V_{\omega} = \frac{1}{4\pi} \iiint \omega(r') \times \frac{r-r'}{|r-r'|^3} dx' dy' dz'$$

Consider the far field effects of a σ and ω distribution



To simplify the derivation, recognize that $V = \nabla \phi$ so that using the 2D potential functions for σ and ω are identical to using V_{σ} and V_{ω} . (i.e. it simplifies the math operations)

The 2D potential of a source is

$$\phi_{\sigma} = \frac{1}{2\pi} \iint \sigma(r') \ln(r-r') dx' dy'$$

"

vorticity is

$$\phi_{\omega} = \frac{1}{2\pi} \iint -\omega(r') \operatorname{atan}\left(\frac{z-z'}{x-x'}\right) dx' dy'$$

Thus,

$$\phi = \frac{1}{2\pi} \iint \sigma \ln R - \omega \Theta dx' dz' \quad \text{where } R = \ln(\sqrt{(x-x')^2 + (z-z')^2})$$

$$\Theta = \operatorname{atan}\left(\frac{z-z'}{x-x'}\right)$$

The objective to discover the far field effects when R is large

Strategy: 1) Taylor series about $(x', z') = 0, 0$ 2) Drop higher order terms

$$\ln R = \ln R|_{0,0} + \frac{\partial \ln R}{\partial x'} \Big|_{0,0} x' + \frac{\partial \ln R}{\partial z'} \Big|_{0,0} z' + \text{Higher order terms}$$

such as $\frac{\partial^2 \ln R}{\partial x'^2} x'^2$
and $\frac{\partial^2 \ln R}{\partial x' \partial z'} x' z'$

Be crafty:

$$\ln R = \ln \sqrt{r^2} = \frac{1}{2} \ln r^2$$

$$\begin{aligned} \frac{\partial \ln R}{\partial x'} &= \frac{1}{2} \frac{\partial \ln r^2}{\partial x'} = \frac{1}{2} \frac{1}{r^2} \frac{\partial r^2}{\partial x'} = \frac{1}{2} \frac{1}{r^2} \frac{\partial}{\partial x'} ((x-x')^2 + (z-z')^2) \\ &= \frac{1}{2} \frac{1}{r^2} 2(x-x') \frac{\partial}{\partial x'} (x-x') = \frac{1}{2} \frac{1}{r^2} 2(x-x')(-1) = -\frac{x-x'}{r^2} \end{aligned}$$

$$\frac{\partial \ln R}{\partial x'} \Big|_{0,0} = -\frac{x}{r^2}$$

And likewise

$$\frac{\partial \ln R}{\partial z'} \Big|_{0,0} = -\frac{z}{r^2}$$

Why about $(x', z') = (0, 0)$?

Gather all perturbations and zoom far away from origin.

So the Taylor series is

$$\ln R \approx \ln r - \frac{x}{r^2} x' - \frac{z}{r^2} z'$$

Similarly, $\Theta = \Theta|_{0,0} + \frac{\partial \Theta}{\partial x'} \Big|_{0,0} x' + \frac{\partial \Theta}{\partial z'} \Big|_{0,0} z' + \text{Higher order terms.}$

$$\begin{aligned} \frac{\partial \arctan \left(\frac{z-z'}{x-x'} \right)}{\partial x'} &= \frac{1}{\left(\frac{z-z'}{x-x'} \right)^2 + 1} \frac{\partial}{\partial x'} \left(\frac{z-z'}{x-x'} \right) = \frac{1}{\left(\frac{z-z'}{x-x'} \right)^2 + 1} (z-z')(-1)(x-x')^2(-1) \\ &= + \frac{z-z'}{(z-z')^2 + (x-x')^2} \end{aligned}$$

$$\Theta \approx \Theta|_{\arctan \frac{z}{x}} + \frac{z}{r^2} x' - \frac{x}{r^2} z'$$

Back to the potential function with the Taylor series substituted

$$\begin{aligned}\phi &= \frac{1}{2\pi} \iint (\sigma \ln R - \omega \theta) dx' dz' \\ &= \frac{1}{2\pi} \iint \left[\sigma \left(\ln r - \frac{x}{r^2} x' - \frac{z}{r^2} z' \right) - \omega \left(\text{atan} \frac{z}{x} + \frac{z}{r^2} x' - \frac{x}{r^2} z' \right) \right] dx' dz'\end{aligned}$$

Notice how this Taylor series simplified the integrals.

$$\begin{aligned}&= \frac{1}{2\pi} \left[\iint \sigma dx' dz' \right] \ln r + \frac{1}{2\pi} \left[\iint (-\sigma x' + \omega z') dx' dz' \right] \frac{x}{r^2} \\ &\quad - \frac{1}{2\pi} \left[\iint \omega dx' dz' \right] \text{atan} \frac{z}{x} + \frac{1}{2\pi} \left[\iint (-\sigma z' - \omega x') dx' dz' \right] \frac{z}{r^2}\end{aligned}$$

Which is

$$\phi = \frac{\Lambda}{2\pi} \ln r - \frac{\Gamma}{2\pi} \theta + \frac{k_x}{2\pi} \frac{x}{r^2} + \frac{k_z}{2\pi} \frac{z}{r^2}$$

$$\text{where } \Lambda = \iint \sigma dx' dz'$$

Sources

and

$$\Gamma = \iint \omega dx' dz'$$

Circulation !!

and

$$k_x = \iint (-\sigma x' + \omega z') dx' dz'$$

$$k_z = \iint (-\sigma z' - \omega x') dx' dz'$$

$$\phi = \frac{\Lambda}{2\pi} \underbrace{\ln \sqrt{x^2 + z^2}}_{\frac{1}{2} \ln r^2} - \frac{\Gamma}{2\pi} \text{atan} \left(\frac{z}{x} \right) + \frac{k_x}{2\pi} \frac{x}{x^2 + z^2} + \frac{k_z}{2\pi} \frac{z}{x^2 + z^2}$$

Velocity is the gradient of potential added to freestream

$$V = V_b + \nabla \phi$$

$$= V_b + \frac{d\phi}{dx} \hat{x} + \frac{d\phi}{dz} \hat{z}$$

This gives (see AFV 2.79 and C.1)

$$V = V_b + \frac{\Lambda}{2\pi} \frac{x \hat{x} + z \hat{z}}{r^2} + \frac{\Gamma}{2\pi} \frac{z \hat{x} - x \hat{z}}{r^2}$$

$$+ \frac{K_x}{2\pi} \frac{(z^2 - x^2) \hat{x} - 2xz \hat{z}}{r^4} + \frac{K_z}{2\pi} \frac{-2xz \hat{x} + (x^2 - z^2) \hat{z}}{r^4}$$

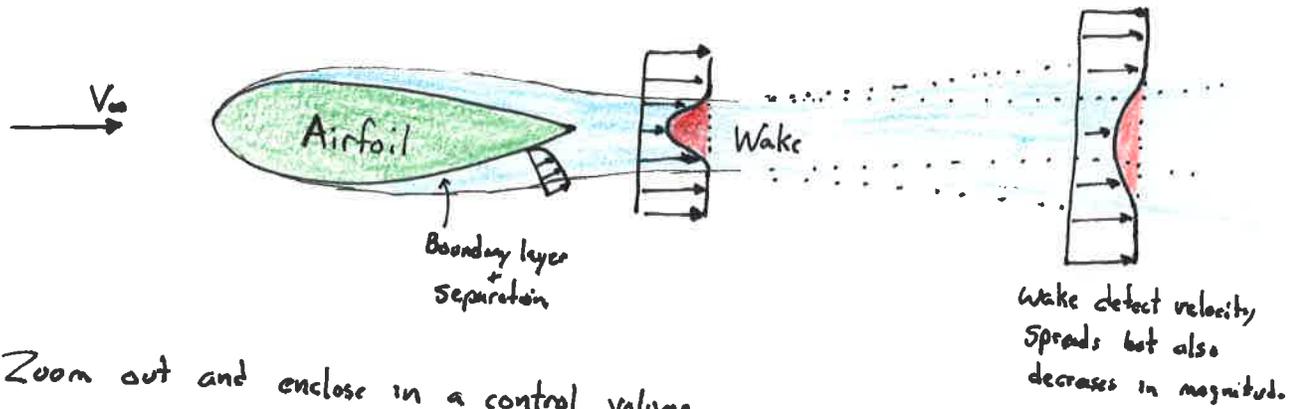
Λ represents source terms 

Γ represents vortex terms 

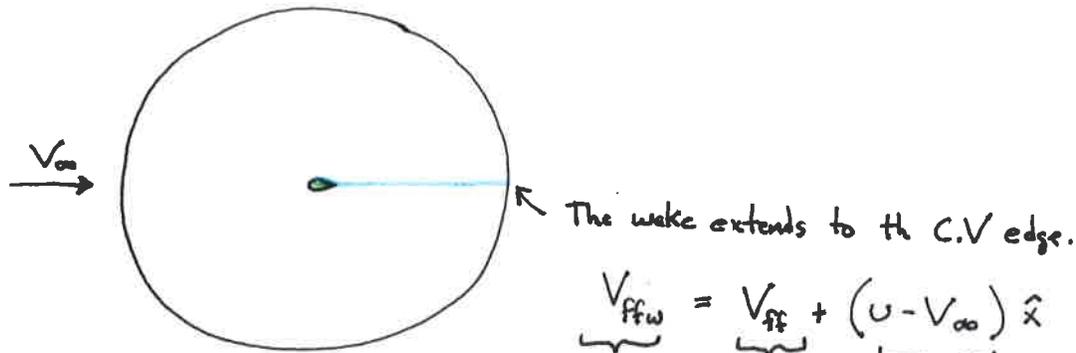
K_x represents doublet terms 

K_z " doublet terms 

Far field Lift and Drag of actual airfoils



Zoom out and enclose in a control volume



$$V_{ffw} = V_{ff} + (u - V_\infty) \hat{x}$$

Velocity at far field in wake = perturbation from airfoil's induced velocity + V_∞ + wake defect

Full C.V.

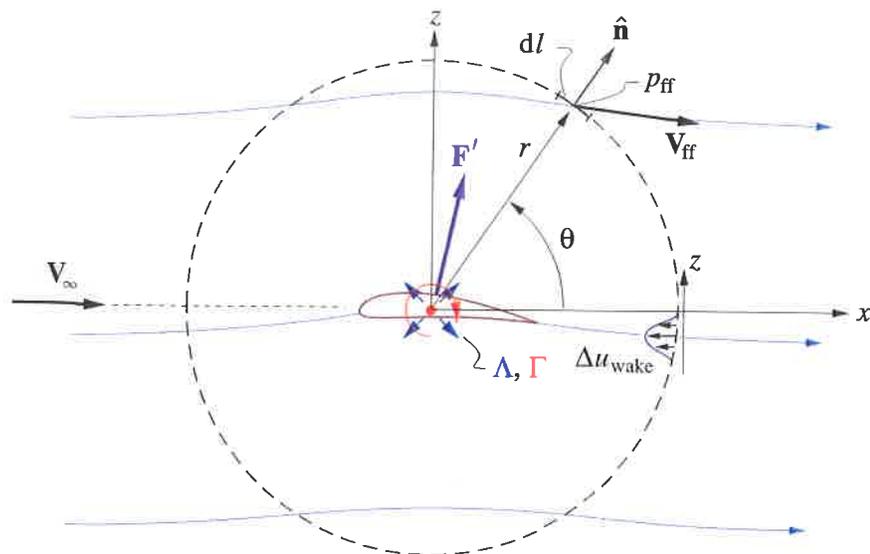


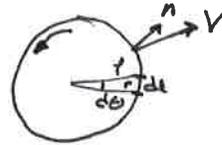
Figure C.2: Outer control volume for calculation of airfoil force. The freestream velocity V_∞ and the Λ, Γ singularities provide the far-field velocity V_{ff} on the contour, which then also provides the pressure p_{ff} via Bernoulli's equation. Terms with the compact wake defect Δu_{wake} will be integrated locally over z .

Mass Conservation

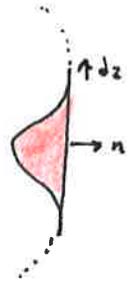
$$\oint \rho (\mathbf{V} \cdot \hat{n}) dl = 0$$

$$= \oint \rho (V_{ffw} \cdot \hat{n}) dl = \int_0^{2\pi} \rho (V_{ffw} \cdot \hat{n}) r d\theta$$

$$= \rho r \int_0^{2\pi} V_{ffw} \cdot \hat{n} d\theta$$



Break V_{ffw} into inviscid part and wake defect (viscous) part.

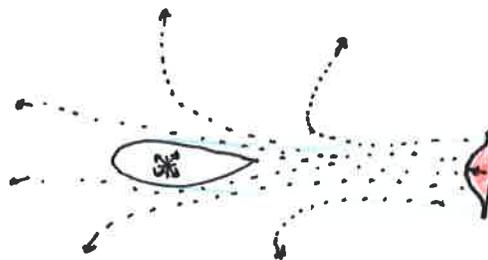


$$= \rho r \int_0^{2\pi} \left(V_{\infty} \cos \theta \hat{r} - \sin \theta \hat{\theta} + \frac{\Lambda}{2\pi r} \hat{r} - \frac{\Gamma}{2\pi r} \hat{\theta} \right) \cdot \hat{n} d\theta + \int \rho (\Delta U_{wake}) \hat{x} \cdot \hat{x} dz$$

$$= \rho r \int_0^{2\pi} \underbrace{\left(V_{\infty} \cos \theta + \frac{\Lambda}{2\pi r} \right)}_{\text{symmetric thus integral } \neq 0} d\theta + \rho \int \underbrace{\Delta U_{wake}}_{\text{Defined as } -\dot{V}'_{wake}, \text{ the wake defect flow rate/span.}} dz$$

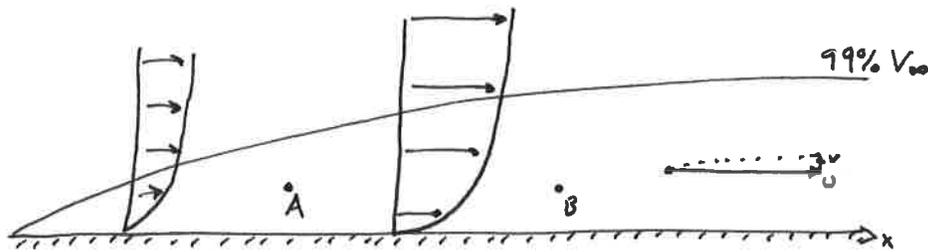
$$= \rho \frac{\Lambda}{2\pi} 2\pi - \rho \dot{V}'_{wake} \Rightarrow \boxed{\dot{V}'_{wake} = \Lambda}$$

The wake behaves as an applied source at the far field, which implies a (small) source at the airfoil.



Alternatively, the wake's defect can be traced back to the farfield. Think of the wake as a negative jet!

Comparison to a Boundary Layer.



When incompressible $\nabla \cdot \mathbf{V} = 0 \Rightarrow \frac{du}{dx} + \frac{dv}{dy} = 0$

At any point (say A),

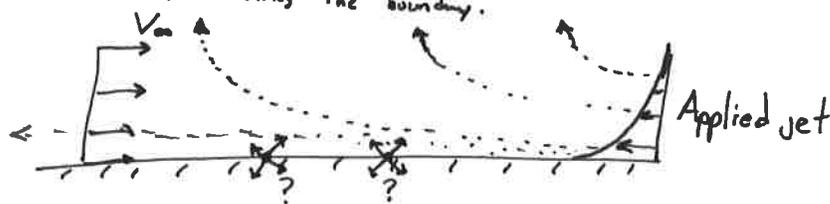
$\frac{du}{dx} < 0$ since from A \rightarrow B, the BL becomes thicker

Thus,

$$\frac{dv}{dy} = -\frac{du}{dx} = -(-) = +$$

There is a slight v velocity, which appears identically to a source!

In bizzaro world, the BL is a freestream with a negative u jet applied at the right. This creates a "source term" along the boundary.



$$\Lambda = \dot{V}'$$

Momentum Conservation

$$F' = - \oint p(V \cdot \hat{n}) V dl + \oint (P_{\infty} - p) \hat{n} dl$$

$$= F' \cdot \hat{x} + F' \cdot \hat{z} = D' + L'$$

$$D' = - \oint p(V \cdot \hat{n})(V \cdot \hat{x}) dl + \oint (P_{\infty} - p)(\hat{n} \cdot \hat{x}) dl$$

$$L' = - \oint p(V \cdot \hat{n})(V \cdot \hat{z}) dl + \oint (P_{\infty} - p)(\hat{n} \cdot \hat{z}) dl$$

Pressure: (Bernoulli)

$$P_{\infty} - P_{ff} = \frac{1}{2} \rho (V_{ff}^2 - V_{\infty}^2) = \frac{1}{2} \rho \left(\left(V_{\infty} (\cos \theta \hat{r} - \sin \theta \hat{\theta}) + \frac{\Lambda}{2\pi r} \hat{r} - \frac{\Gamma}{2\pi r} \hat{\theta} \right)^2 - V_{\infty}^2 \right)$$

$$= \frac{1}{2} \rho \left(\left(V_{\infty} \cos \theta + \frac{\Lambda}{2\pi r} \right)^2 + \left(-V_{\infty} \sin \theta - \frac{\Gamma}{2\pi r} \right)^2 - V_{\infty}^2 \right)$$

$$= \frac{1}{2} \rho \left(\underbrace{V_{\infty}^2 \cos^2 \theta + 2V_{\infty} \cos \theta \frac{\Lambda}{2\pi r} + V_{\infty}^2 \sin^2 \theta}_{V_{\infty}^2 + \left(\frac{\Lambda}{2\pi r}\right)^2} + 2V_{\infty} \sin \theta \frac{\Gamma}{2\pi r} + \left(\frac{\Gamma}{2\pi r}\right)^2 - V_{\infty}^2 \right)$$

$$= \frac{1}{2} \rho \left(2V_{\infty} \cos \theta \frac{\Lambda}{2\pi r} + 2V_{\infty} \sin \theta \frac{\Gamma}{2\pi r} + \frac{\Lambda^2 + \Gamma^2}{(2\pi r)^2} \right)$$

Lift

$$L' = - \oint p(\mathbf{v} \cdot \hat{\mathbf{n}})(\mathbf{v} \cdot \hat{\mathbf{z}}) dl + \int (p_\infty - p)(\hat{\mathbf{n}} \cdot \hat{\mathbf{z}}) dl$$

$$= - \int_0^{2\pi} p r \left(\frac{V_\infty \Lambda}{2\pi r} \sin\theta \cos\theta - \frac{V_\infty \Gamma}{2\pi r} \cos^2\theta + \frac{\Lambda^2}{(2\pi r)^2} \sin\theta - \frac{\Lambda \Gamma}{(2\pi r)^2} \cos\theta \right) d\theta$$

$$- p \int \Delta u_{\text{wake}} \left(\frac{\Lambda}{2\pi r} \sin\theta - \frac{\Gamma}{2\pi r} \cos\theta \right) dz$$

$$+ r \int_0^{2\pi} \frac{1}{2} p \left(2V_\infty \frac{\Lambda}{2\pi r} \cos\theta + 2V_\infty \frac{\Lambda}{2\pi r} \sin\theta + \frac{\Lambda^2 + \Gamma^2}{(2\pi r)^2} \right) \sin\theta d\theta$$

$$= \underbrace{\frac{\rho V_\infty \Gamma}{2} - \frac{\rho \dot{V} \Gamma}{2\pi r}}_{\text{from momentum flow}} + \underbrace{\frac{\rho V_\infty \Gamma}{2}}_{\text{from pressure}}$$

$$L' = \rho V_\infty \Gamma - \frac{\rho \dot{V} \Gamma}{2\pi r}$$

or when $\frac{\dot{V}}{2\pi r} \ll V_\infty$

$$L' = \rho V_\infty \Gamma$$

Kutta Joukowski Theorem

Lift only depends on vorticity/vortex in flow, circulation

Drag

$$D' = \underbrace{-\int p(V \cdot \hat{n})(V \cdot \hat{x}) dl}_{\text{mom' flow}} + \underbrace{\int (p_\infty - p)(\hat{n} \cdot \hat{x}) dl}_{\text{pressure}}$$

mom' flow

$$\begin{aligned} -\int p(V \cdot \hat{n})(V \cdot \hat{x}) dl &= -\rho \int (V_\infty \cos \theta + \frac{\Lambda}{2\pi r} + \Delta U_{\text{wake}}) \left(V_\infty + \frac{\Lambda}{2\pi r} \cos \theta + \frac{\Gamma}{2\pi r} \sin \theta + \Delta U_{\text{wake}} \right) dl \\ &= -\rho r \int_0^{2\pi} \left(V_\infty^2 \cos \theta + \frac{V_\infty \Lambda}{2\pi r} \cos^2 \theta + \frac{V_\infty \Gamma}{2\pi r} \sin \theta \cos \theta + V_\infty \cos \theta \Delta U_{\text{wake}} \right. \\ &\quad \left. + \frac{V_\infty \Lambda}{2\pi r} + \frac{\Lambda^2}{(2\pi r)^2} \cos^2 \theta + \frac{\Gamma \Lambda}{(2\pi r)^2} \sin \theta + \frac{\Lambda}{2\pi r} \Delta U_{\text{wake}} + V_\infty \Delta U_{\text{wake}} + \right. \\ &\quad \left. + \frac{\Lambda}{2\pi r} \cos \theta \Delta U_{\text{wake}} + \frac{\Gamma}{2\pi r} \sin \theta \Delta U_{\text{wake}} + \Delta U_{\text{wake}}^2 \right) d\theta \end{aligned}$$

Simplify integrals $\rightarrow 0$ and separate ΔU_{wake} terms while converting to dz integral ($dz = d\theta r$)

$$\begin{aligned} &= -\rho r \int_0^{2\pi} \left(\frac{V_\infty \Lambda}{2\pi r} \cos^2 \theta + \frac{V_\infty \Lambda}{2\pi r} \right) d\theta - \rho \int \left(V_\infty \Delta U_{\text{wake}} \cos \theta + \frac{2\Lambda}{2\pi r} \Delta U_{\text{wake}} + V_\infty \Delta U_{\text{wake}} \right. \\ &\quad \left. + \frac{\Lambda}{2\pi r} \sin \theta \Delta U_{\text{wake}} + \Delta U_{\text{wake}}^2 \right) dz \\ &= -\rho r \frac{V_\infty \Lambda}{2\pi r} \pi - \rho r \frac{V_\infty \Lambda}{2\pi r} 2\pi - \rho \int (V_\infty \Delta U_{\text{wake}} + \Delta U_{\text{wake}}^2) dz - \rho \int \left(\frac{\Lambda}{\pi r} \Delta U_{\text{wake}} \right) dz \\ &= -\frac{\rho V_\infty \Lambda}{2} - \rho V_\infty \Lambda - \rho \int (V_\infty + \Delta U_{\text{wake}})(\Delta U_{\text{wake}}) dz - \rho \int V_\infty \Delta U_{\text{wake}} dz \\ &\quad - \rho \int \frac{\Lambda}{2\pi r} \Delta U_{\text{wake}} dz - \rho \int V_\infty \Delta U_{\text{wake}} dz \\ &\quad \underbrace{-\rho V_\infty \dot{V}' = -\Lambda \rho V_\infty} \end{aligned}$$

$$= -\frac{\rho V_\infty \Lambda}{2} + \frac{\rho \Lambda}{2\pi r} \dot{V}' + P_\infty$$

where P is the momentum defect

$$\begin{aligned} P_\infty &\equiv \rho \int (V_\infty + \Delta U_{\text{wake}})(-\Delta U_{\text{wake}}) dz \\ &= \int \rho U (V_\infty - U) dz \end{aligned}$$

pressure.

$$\begin{aligned} \int (P_\infty - P)(\hat{n} \cdot \hat{x}) dl &= \int_0^{2\pi} \frac{1}{2} \rho (V_{\text{ff}}^2 - V_\infty^2) \cos \theta r d\theta \\ &= \frac{1}{2} \rho r \int_0^{2\pi} \left(2V_\infty \frac{\Lambda}{2\pi r} \cos \theta + 2V_\infty \frac{\Gamma}{2\pi r} \sin \theta + \frac{\Lambda^2 + \Gamma^2}{(2\pi r)^2} \right) \cos \theta d\theta \\ &= \frac{1}{2} \rho r 2V_\infty \frac{\Lambda}{2\pi r} \pi = \frac{\rho V_\infty \Lambda}{2} \end{aligned}$$

Combining gives

$$D' = P_\infty + \frac{\rho \Lambda}{\pi r} \dot{v}'$$

$$D' = P_\infty + \frac{\rho \dot{v}'^2}{\pi r}$$

for reasonable $\dot{v}' < 1$

$$D' = P_\infty$$

Also, since $\Delta U_{\text{wake}} \rightarrow 0$ as the wake spreads

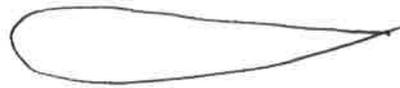
$$\rho V_\infty \dot{v}' = \rho V_\infty \int (-\Delta U_{\text{wake}}) dz \iff \underbrace{\rho \int (V_{\text{wake}} + \Delta U_{\text{wake}}) (-\Delta U_{\text{wake}}) dz}_{\text{definition of } P_\infty}$$

$$D' = \rho V_\infty \dot{v}'$$

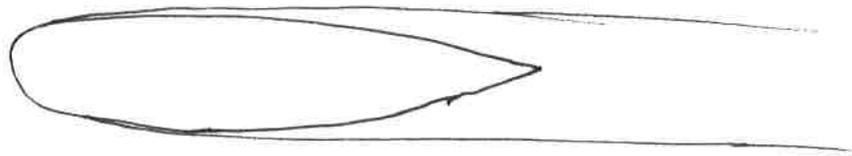
$$D' = \rho V_\infty \Lambda$$

Recap

$$L' = \rho V_{\infty} \Gamma$$
$$D' = \rho V_{\infty} \Lambda$$



$$\Lambda = \int \lambda(x) dx$$



$$\Lambda = \int \lambda(x) dx \neq 0$$