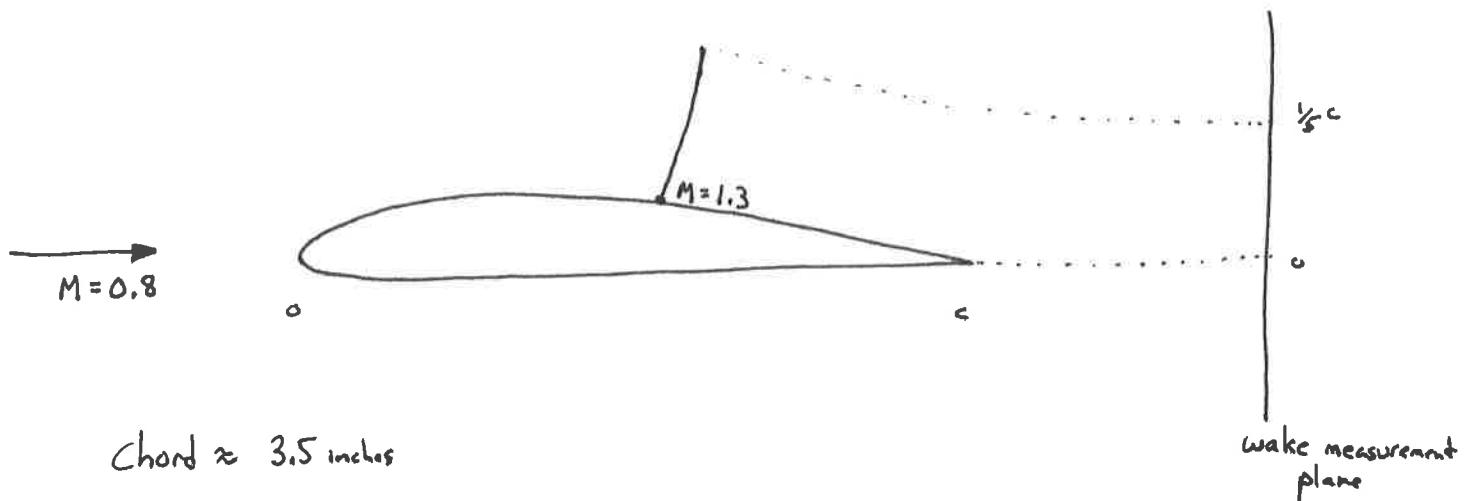


Lesson 16

Full Potential

Ex: Estimate the wave drag for the following airfoil. Assume the incoming Mach # varies linearly from the top to the bottom of the shock.



At measurement plane, the streamtube containing the shock is 0.7 inches.

$$\frac{z}{c} = \frac{0.7}{3.5} = \frac{1}{5}$$

$$C_d^{\text{wave}} = 2 \int_0^{1/5} \left(1 - \frac{V_{\text{wake}}}{V_{\infty}}\right) \frac{dz}{c} = 2 \int_0^{1/5} \left(1 - \frac{V_{\text{wake}}}{V_{\infty}}\right) \frac{dz}{c}$$

$$\frac{V_{\text{wake}}}{V_{\infty}} \approx 1 - \frac{1}{M_{\infty}} \frac{(M_1 - 1)^3}{1 + 2\gamma(M_1 - 1)}$$



$$M_1 = 1.3 - \frac{(1.3 - 1.0)}{\frac{1}{5}} z = 1.3 - 1.5 z$$

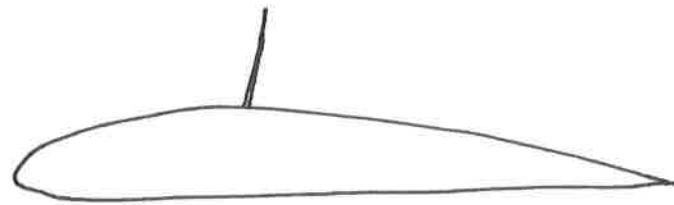
$$1 - \frac{V_{\text{wake}}}{V_{\infty}} = \frac{1}{M_{\infty}} \frac{((1.3 - 1.5z) - 1)^3}{1 + 2\gamma((1.3 - 1.5z) - 1)}$$

$$C_d = 2 \int_0^{1/5} \frac{1}{0.8} \frac{((1.3 - 1.5z) - 1)^3}{1 + 2 \cdot 1.4((1.3 - 1.5z) - 1)} dz$$

$$= 0.0020 \quad 20 \text{ counts}$$

$M_{DD} = 0.8$  for this completely made up condition!

Q:



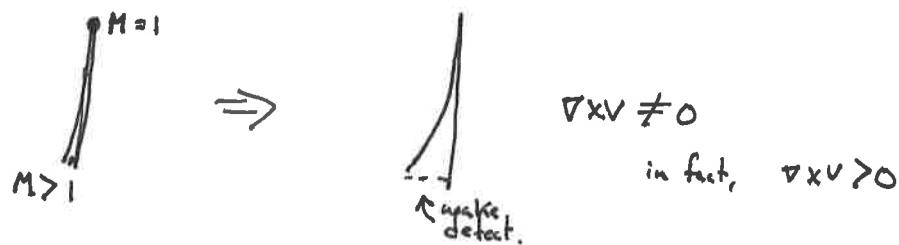
Is a shock isentropic? No.

Is a shock irrotational?

$$\nabla \times V \quad \Rightarrow \quad \begin{bmatrix} \nearrow \\ \searrow \end{bmatrix}$$

Normal shock of constant strength: Yes.

What about the airfoil's shock?



"Mostly" irrotational for weak shocks

Potential functions  $V = \nabla \phi$  are irrotational.

A special method is required to determine the wave drag for compressible potential flows. See FVA 8.4.3

# Full Potential: Compressible flows.

Mass Continuity equation  $\frac{d\rho}{dt} + \nabla \cdot (\rho V) = 0$

When steady state  $\nabla \cdot (\rho V) = 0$

Represent  $V$  with  $\nabla \phi$  since relatively "weak" shocks are relatively irrotational

$$\underbrace{\nabla \cdot (\rho \nabla \phi)}_0 = 0$$

This is just the mass continuity equation.

How can density be represented / modeled?

- Constant incompressible  $\nabla \cdot (\rho \nabla \phi) = \rho \nabla \cdot \nabla \phi = \rho \nabla^2 \phi = 0$
- Isentropic flow
- Something else...

For isentropic flow, AFV section 8.2.2 provides  $\frac{\rho}{\rho_\infty}$  and  $\frac{P}{P_\infty}$

$$\frac{\rho}{\rho_\infty} = \left[ 1 + \frac{\gamma-1}{2} M_\infty^2 \left( 1 - \frac{V^2}{V_\infty^2} \right) \right]^{\frac{1}{\gamma-1}}$$

$$\text{Substitute } V = \nabla \phi \Rightarrow V^2 = U^2 + V^2 + \omega^2 = V \cdot V = \nabla \phi \cdot \nabla \phi$$

$$\frac{\rho}{\rho_\infty} = 1 + \frac{\gamma-1}{2} M_\infty^2 \left( 1 - \frac{\nabla \phi \cdot \nabla \phi}{V_\infty^2} \right)^{\frac{1}{\gamma-1}}$$

Boundary Conditions

$$V \cdot n = 0 \text{ on the surface} \Rightarrow \nabla \phi \cdot n = 0$$

Also need to account for the branch cut due to circulation

$$\Delta \phi = \Gamma$$



# Small Disturbance Compressible Flows

Given a flow in the  $\hat{x}$  direction:

$$\mathbf{V} = V_{\infty} \hat{x}$$

↑ Vector      ↑ Scalar

$$\mathbf{V} = V_{\infty} \hat{x} + u V_{\infty} \hat{x} + v V_{\infty} \hat{y} + w V_{\infty} \hat{z} = V_{\infty} \left[ (1+u) \hat{x} + v \hat{y} + w \hat{z} \right]$$

$$V^2 = \mathbf{V} \cdot \mathbf{V} = V_{\infty}^2 \left( (1+u)^2 + v^2 + w^2 \right) = V_{\infty}^2 \left( 1 + 2u + u^2 + v^2 + w^2 \right)$$

Substitute into:

$$\begin{aligned} a^2 &= a_{\infty}^2 \left( 1 + \frac{\gamma-1}{2} M_{\infty}^2 \left( 1 - \frac{V^2}{V_{\infty}^2} \right) \right) \\ &= a_{\infty}^2 \left( 1 + \frac{\gamma-1}{2} M_{\infty}^2 \left( 1 - (1 + 2u + u^2 + v^2 + w^2) \right) \right) \\ &= a_{\infty}^2 \left( 1 + \frac{\gamma-1}{2} M_{\infty}^2 (-2u - u^2 - v^2 - w^2) \right) \\ &= a_{\infty}^2 \left( 1 - \frac{\gamma-1}{2} M_{\infty}^2 (2u + u^2 + v^2 + w^2) \right) \end{aligned}$$

Solve for

$$M^2 = \frac{V^2}{a^2} \quad \text{and} \quad \frac{\rho}{\rho_{\infty}} \quad \text{and} \quad \frac{P}{P_{\infty}}$$

$$\begin{aligned}
a^2 &= a_{\infty}^2 \left( 1 + \frac{\gamma-1}{2} M_{\infty}^2 \left( 1 - \frac{V^2}{V_{\infty}^2} \right) \right) \\
&= a_{\infty}^2 \left( 1 + \frac{\gamma-1}{2} M_{\infty}^2 \left( 1 - \frac{V_{\infty}^2}{V_{\infty}^2} (1 + 2u + u^2 + v^2 + w^2) \right) \right) \\
&= a_{\infty}^2 \left( 1 + \frac{\gamma-1}{2} M_{\infty}^2 (2u + u^2 + v^2 + w^2) \right) \quad E_8 \ 8.34 \\
&= a_{\infty}^2 \left( 1 - (\gamma-1) M_{\infty}^2 \left( u + \frac{1}{2} u^2 + \frac{1}{2} v^2 + \frac{1}{2} w^2 \right) \right)
\end{aligned}$$

$$\begin{aligned}
M^2 &= \frac{V^2}{a^2} = \frac{V_{\infty}^2 (1 + 2u + u^2 + v^2 + w^2)}{a_{\infty}^2 (1 - (\gamma-1) M_{\infty}^2 (u + \frac{1}{2} u^2 + \frac{1}{2} v^2 + \frac{1}{2} w^2))} \\
&= M_{\infty}^2 \frac{(1 + 2u + u^2 + v^2 + w^2)}{1 - (\gamma-1) M_{\infty}^2 (u + \frac{1}{2} u^2 + \frac{1}{2} v^2 + \frac{1}{2} w^2)}
\end{aligned}$$

$$\begin{aligned}
\frac{r}{P_{\infty}} &= \left[ 1 + \frac{\gamma-1}{2} M_{\infty}^2 \left( 1 - \frac{V^2}{V_{\infty}^2} \right) \right]^{\frac{1}{\gamma-1}} \\
&= \left[ 1 + \frac{\gamma-1}{2} M_{\infty}^2 \left( 1 - \frac{V_{\infty}^2 (1 + 2u + u^2 + v^2 + w^2)}{V_{\infty}^2} \right) \right]^{\frac{1}{\gamma-1}} \\
&= \left[ 1 + \frac{\gamma-1}{2} M_{\infty}^2 (2u + u^2 + v^2 + w^2) \right]^{\frac{1}{\gamma-1}} \\
&= \left[ 1 - (\gamma-1) M_{\infty}^2 \left( u + \frac{1}{2} u^2 + \frac{1}{2} v^2 + \frac{1}{2} w^2 \right) \right]^{\frac{1}{\gamma-1}}
\end{aligned}$$

If  $f = (1-\epsilon)^b$ , what is the Taylor series expansion of  $f$ ? about  $\epsilon=0$

$$f \approx f_0 + \frac{df}{d\epsilon} \epsilon + \frac{1}{2} \frac{d^2f}{d\epsilon^2} \epsilon^2 + \dots$$

$$\frac{df}{d\epsilon} = \frac{d((1-\epsilon)^b)}{d(1-\epsilon)} \cdot \frac{d(1-\epsilon)}{d\epsilon} = b(1-\epsilon)^{b-1} \cdot (-1) \approx -b$$

$$\frac{d^2f}{d\epsilon^2} = \text{by a similar process} = b \cdot (b-1)(1-\epsilon)^{b-2} \approx b \cdot (b-1)$$

$$f \approx 1 + -b\epsilon + b(b-1)\epsilon^2$$

for the limited case of  $b=-1$

$$f \approx 1 + \epsilon + \epsilon^2$$

$$\begin{aligned} \text{For } M^2 = \frac{V^2}{a^2} &= M_\infty^2 \underbrace{(1+2u+u^2+v^2+w^2)}_C \underbrace{\left(1 - (\gamma-1)M_\infty^2 \left(u + \frac{1}{2}(u^2+v^2+w^2)\right)\right)^{-1}}_{(1-\epsilon)^{-1}} \Rightarrow \epsilon = \frac{\gamma-1}{M_\infty^2}(\dots) \\ &\approx C(1+\epsilon+\epsilon^2) = C + C \frac{\gamma-1}{M_\infty^2} \left(u + \frac{1}{2}(u^2+v^2+w^2)\right) + C \frac{(\gamma-1)^2}{M_\infty^4} \left(u + \frac{1}{2}(u^2+v^2+w^2)\right)^2 \end{aligned}$$

collect terms of similar order (i.e.  $u^2, v^2, uv, \dots$ )

$$\begin{aligned} \frac{M^2}{M_\infty^2} &\approx (1+2u+u^2+v^2+w^2) + (\gamma-1)(1+2u+u^2+v^2+w^2) \left(u + \frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2}\right) \\ &\quad + (1+2u+u^2+v^2+w^2)(\gamma-1)^2 M_\infty^2 \left(u^2 + \frac{u^3}{2} + \frac{uv^2}{2} + \frac{vw^2}{2} + \frac{u^3}{2} + \frac{u^4}{4} + \frac{u^2v^2}{4} + \frac{v^3w^2}{4}\right. \\ &\quad \left. + \frac{uv^2}{2} + \frac{u^2v^2}{4} + \frac{v^4}{4} + \frac{v^2w^2}{4} + \frac{uw^2}{2} + \frac{u^2w^2}{4} + \frac{w^3v^2}{4} + \frac{w^4}{4}\right) \end{aligned}$$

What did we do here?

Convert powers to polynomials!

This appears to be high order terms, but it contains a hidden "u".  $u^3 + \frac{uv^2}{2} + \frac{vw^2}{2} + \frac{u^3}{2} + \frac{uw^2}{2} = u$

• Isentropic

$$\frac{\rho}{\rho_\infty} = \left( \frac{a^2}{a_{\infty}^2} \right)^{\frac{1}{Y-1}} = \left( \text{from prev page} \right)^{\frac{1}{Y-1}} = \left[ 1 - \frac{Y-1}{2} M_\infty^2 \left( U + \frac{1}{2}(U^2 + V^2 + W^2) \right) \right]^{\frac{1}{Y-1}}$$

$$\text{use } (1-\epsilon)^b \approx 1 - b\epsilon + \frac{1}{2}b(b-1)\epsilon^2$$

$$\frac{\rho}{\rho_\infty} = 1 - \frac{1}{Y-1} \frac{Y-1}{2} M_\infty^2 \left( U + \frac{1}{2}U^2 + \frac{1}{2}V^2 + \frac{1}{2}W^2 \right) + \frac{1}{2} \frac{1}{Y-1} \left( \frac{1}{Y-1} - 1 \right) (Y-1)^2 M_\infty^4 \left( U + \frac{1}{2}U^2 + \frac{1}{2}V^2 + \frac{1}{2}W^2 \right)^2$$

$$\frac{1}{Y-1} - \frac{Y-1}{2} = -\frac{1}{2}$$

• Substitute into mass continuity  $\nabla \cdot (\rho V) = 0$

$$\nabla \cdot \left( \rho_\infty \frac{\rho}{\rho_\infty} V_\infty [(1+U)\hat{x} + V\hat{y} + W\hat{z}] \right) = 0$$

thus

$$\nabla \cdot (A\hat{x} + B\hat{y} + C\hat{z}) = \frac{d}{dx}(A) + \frac{d}{dy}(B) + \frac{d}{dz}(C) = 0$$

•  $\hat{x}$  term

$$\frac{\rho}{\rho_\infty} = 1 - M_\infty^2 \left( U + \frac{1}{2}U^2 + \frac{1}{2}V^2 + \frac{1}{2}W^2 \right) - \frac{Y}{2} M_\infty^4 \underbrace{\left( U + \frac{1}{2}U^2 + \frac{1}{2}V^2 + \frac{1}{2}W^2 \right)^2}_{\bullet + \cancel{\bullet} + \cancel{\bullet} + \cancel{\bullet}}$$

$$\bullet + \cancel{\bullet} + \cancel{\bullet} + \cancel{\bullet} \\ U^2 + U^3 + UV^2 + UW^2 + \dots$$

Tricky here.  
 $U^2 + V^2 + W^2 \neq 1$

Multiply by  $(1+U)$  and collect terms.

$$\left( \frac{\rho}{\rho_\infty} \right) (1+U) \approx 1 + U +$$

As computed with MathCAD's CAS.

$$\frac{P}{P_\infty} = 1 - M_\infty^2 U - \frac{M_\infty^2}{2} U^2 - \frac{\gamma}{2} M_\infty^4 U^2 - M_\infty^2 \left( \frac{1}{2} U^2 + \frac{1}{2} \omega^2 \right) + \dots$$

$$\begin{aligned}\frac{P}{P_\infty}(1+U) &= 1 + U - M_\infty^2 U - \frac{3}{2} M_\infty^2 U^2 - \frac{\gamma}{2} M_\infty^4 U^2 - M_\infty^2 \left( \frac{V^2}{2} + \frac{\omega^2}{2} \right) \\ &= 1 + (1 - M_\infty^2)U - \underbrace{\frac{1}{2} M_\infty^2 (3 + \gamma M_\infty^2)}_{\text{not the same as the book!!}} U^2 - M_\infty^2 \left( \frac{V^2}{2} + \frac{\omega^2}{2} \right)\end{aligned}$$

$$\frac{P}{P_\infty} V = V - M_\infty^2 UV \quad \text{similar with } \frac{P}{P_\infty} \omega = \omega - M_\infty^2 UW$$

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Governing Egu.

$$\frac{d}{dx} \left[ (1 - M_\infty^2) U - \frac{1}{2} M_\infty^2 ((3 + \gamma M_\infty^2) U^2 + V^2 + \omega^2) \right] + \frac{d}{dy} (V - M_\infty^2 UV) + \frac{d}{dz} (\omega - M_\infty^2 UW) = 0$$

BCs

$$V \cdot n = 0 \Rightarrow (1+U)n_x + Vn_y + \omega n_z = 0$$