

Lesson 17

Linearized Compressible Flow

Prandtl-Glauert

## Perturbation Velocities

$$U = V_{\infty} + U'$$

$$V = V'$$

$$W = W'$$

$$\text{and } \frac{V'}{a} \ll 1$$

$$\frac{W'}{c} \ll 1$$

$$\left(1 - \frac{U^2}{a^2}\right) \frac{dU}{dx} + \frac{dV}{dy} + \frac{dW}{dz} = 0$$

Recall from a previous lesson that the full isentropic  $\frac{a}{a_{\infty}}$  can be decomposed into a Taylor series (binomial eqn).

$$\frac{a^2}{a_{\infty}^2} = 1 + \frac{\gamma-1}{2} M_{\infty}^2 \left( 2 \frac{U'}{U_{\infty}} + \frac{U'^2 + V'^2 + W'^2}{U_{\infty}^2} \right)$$

Such that

$$\begin{aligned} \left(1 - \frac{U^2}{a^2}\right) &= 1 - \frac{U_{\infty}^2 + 2U'U_{\infty} + U'^2}{U_{\infty}^2} M_{\infty}^2 \frac{a_{\infty}^2}{a^2} \\ &= 1 - M_{\infty}^2 \left( 1 + \frac{2U'}{U_{\infty}} \left( 1 + \frac{\gamma-1}{2} M_{\infty}^2 \right) \right) \end{aligned}$$

# Compressible Subsonic Flow

Continuity:  $\nabla \cdot (\rho \mathbf{V}) = 0$

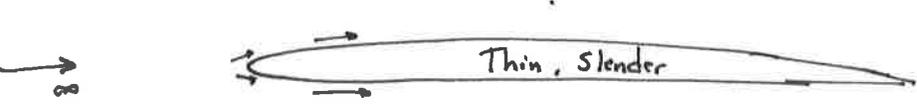
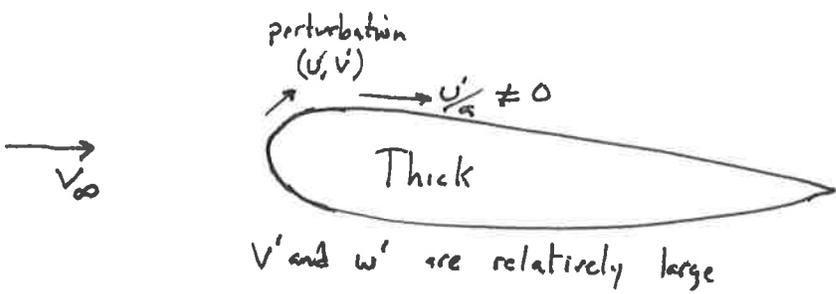
Momentum:  $\nabla \cdot (\rho \mathbf{V} \mathbf{V}^T) + \nabla p = 0$

Isentropic:  $\frac{dp}{\rho} = a^2$

Irrrotational:  $\nabla \times \mathbf{V} = 0$

Combine these to yield

$$\underbrace{\left(1 - \frac{U^2}{a^2}\right) \frac{dU}{dx} + \left(1 - \frac{V^2}{a^2}\right) \frac{dV}{dy} + \left(1 - \frac{W^2}{a^2}\right) \frac{dW}{dz}}_{\text{mixed velocities}} - 2 \frac{UV}{a^2} \frac{dU}{dy} - 2 \frac{UW}{a^2} \frac{dV}{dz} - 2 \frac{WU}{a^2} \frac{dW}{dx} = 0$$



For a slender airfoil, the  $v'$  and  $w'$  perturbations are small with respect to  $a^2$

The compressible flow eqn. is now.

$$(1 - M_\infty^2) \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = M_\infty^2 \left(1 + \frac{\gamma-1}{2} M_\infty^2\right) \frac{2u'}{U_\infty} \frac{du}{dx}$$

Since  $u = V_\infty + u'$ , the derivative of a constant is zero. Thus, the above equation is also written in terms of perturbations  $u', v', w'$ .

$$(1 - M_\infty^2) \frac{du'}{dx} + \frac{dv'}{dy} + \frac{dw'}{dz} = \frac{2}{U_\infty} \left(1 + \frac{\gamma-1}{2} M_\infty^2\right) M_\infty^2 u' \frac{du'}{dx}$$

This is a form of the Transonic Small Disturbance equation.  
TSD.

This is also written in terms of a potential function

$$V = \nabla \phi \Rightarrow u' = \phi_x \quad v' = \phi_y \quad w' = \phi_z$$

$$(1 - M_\infty^2) \phi_{xx} + \phi_{yy} + \phi_{zz} = \frac{2}{U_\infty} \left(1 + \frac{\gamma-1}{2} M_\infty^2\right) M_\infty^2 \phi_x \phi_{xx}$$

Why use a potential function?

- Single field variable,  $\phi$ , contains all physics
- Simple flow properties and streamlines.

Review: Incompressible Flow, Irrotational

$$\nabla \cdot \mathbf{V} = 0 \quad (\text{continuity})$$

and

$$\mathbf{V} = \nabla \phi \quad (\text{potential})$$

$$\nabla \cdot \nabla \phi = 0$$

$$\boxed{\nabla^2 \phi = 0}$$

Laplace Equation from PDE.

In 3D cartesian coordinates

$$\boxed{\phi_{xx} + \phi_{yy} + \phi_{zz} = 0}$$

Pressure Coefficient

$$C_p = 1 - \frac{V^2}{V_\infty^2} \quad \text{for a subsonic flow}$$

# Prandtl Glauert

Simplify the TSD equation with an assumption that  $M_{\infty}^2 \phi_x \phi_{xx} \ll 1$   
 $\leftarrow \approx 0$

$$(1 - M_{\infty}^2) \phi_{xx} + \phi_{yy} + \phi_{zz} = 0$$

This is now a linear PDE!

Transformation:

$$x' = \frac{x}{\beta}$$

$$\text{where } \beta = \sqrt{1 - M_{\infty}^2}$$

$$y' = y$$

$$z' = z$$

$$\Rightarrow \frac{d}{dx} = \frac{d}{dx'} \frac{dx'}{dx} = \frac{1}{\beta} \frac{d}{dx'}$$
$$\frac{d}{dy} = \frac{d}{dy'} \frac{dy'}{dy} = \frac{d}{dy'}$$
$$\frac{d}{dz} = \frac{d}{dz'} \frac{dz'}{dz} = \frac{d}{dz'}$$

2nd derivatives

$$\frac{d^2}{dx^2} = \frac{d}{dx} \left( \frac{d}{dx} \right) = \frac{1}{\beta^2} \frac{d}{dx'} \left( \frac{d}{dx'} \right) = \frac{1}{\beta^2} \frac{d^2}{dx'^2}$$

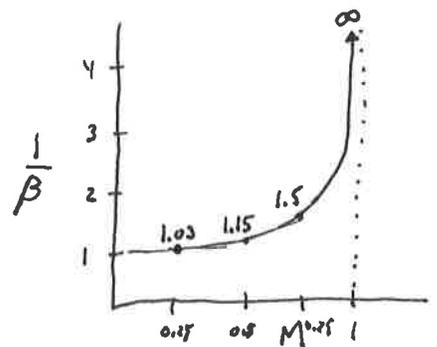
$$\frac{d^2}{dy^2} = \frac{d}{dy} \left( \frac{d}{dy} \right) = \frac{d^2}{dy'^2}$$

plug into p-6 equation

$$(1 - M_{\infty}^2) \frac{1}{\beta^2} \phi_{x'x'} + \phi_{y'y'} + \phi_{z'z'} = 0$$

$$\text{But } \frac{1}{\beta^2} = \frac{1}{1 - M_{\infty}^2}$$

$$\phi_{x'x'} + \phi_{y'y'} + \phi_{z'z'} = 0$$

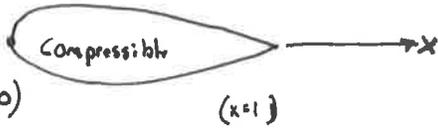


Stretched in the x direction, a p-6 compressible flow is exactly identical to an incompressible flow!  
Amazing!

Visually (stretch  $x$ )

$$M = 0.75$$

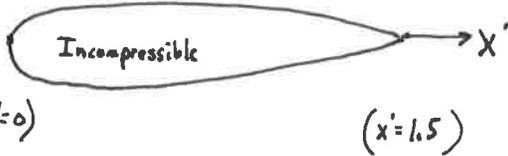
$$(x = 0)$$



$$\beta = \sqrt{1 - M_{\infty}^2} = \sqrt{1 - 0.75^2} \approx 0.67$$

$$M = 0$$

$$(x' = 0)$$



Same thickness (absolute) but effective  $\frac{t'}{c} = \beta \frac{t}{c}$

Alternatively (2<sup>nd</sup> way of a transform)

$$x'' = x$$

$$y'' = \beta y$$

$$z'' = \beta z$$

$$\frac{d}{dx} = \frac{d}{dx''}$$

$$\frac{d}{dy} = \frac{d}{dy''} \frac{dy''}{dy} = \beta \frac{d}{dy''}$$

$$\frac{d}{dz} = \frac{d}{dz''} \frac{dz''}{dz} = \beta \frac{d}{dz''}$$

$$\frac{d^2}{dx^2} = \frac{d^2}{dx''^2}$$

$$\frac{d^2}{dy^2} = \beta^2 \frac{d^2}{dy''^2}$$

$$\frac{d^2}{dz^2} = \beta^2 \frac{d^2}{dz''^2}$$

Subs into  $(1 - M_{\infty}^2) \phi_{xx} + \phi_{yy} + \phi_{zz} = 0$

$$\beta^2 \phi_{xx''} + \beta^2 \phi_{y''y''} + \beta^2 \phi_{z''z''} = 0$$

$$\phi_{xx''} + \phi_{y''y''} + \phi_{z''z''} = 0$$

Stretch  $y$  and  $z$

$$M = 0.75$$

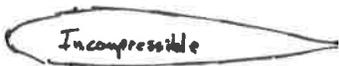
$$x=0$$



$$x=1$$

$$\frac{t}{c}$$

$$M = 0$$



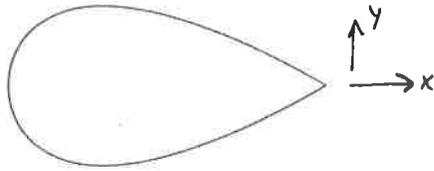
$$\frac{t'}{c} = \beta \frac{t}{c}$$

## Visual Compressibility via PG

WARNING: Please don't think that these streamline accurately represent the NACA 0050 at Mach 0.90. This analysis is purely considering the visual effect of compressibility. This process occurs for thinner sections, but the transformation is not as dramatic.

NACA 0050  
Thickness Exaggerated for effect  
Mach = 0.9

$$\beta = \sqrt{1 - M_\infty^2}$$



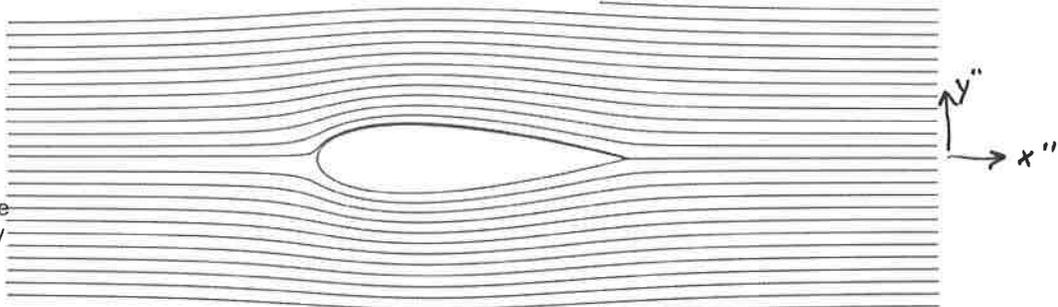
WARNING: PG requires small v and w perturbations. This NACA 0050 fails that requirement.

Transform with PG  
Beta = 0.43

$$y'' = 0.43 y$$

$$x'' = x$$

Solve incompressible  
flow about effectively  
an NACA 0022



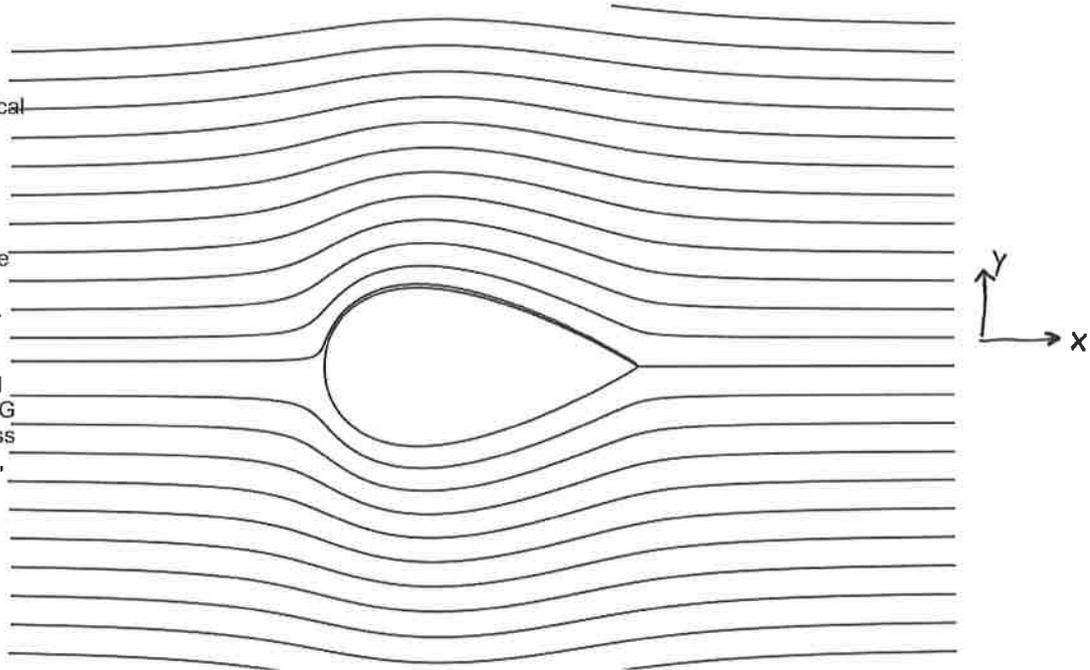
Rescale back to physical  
dimensions

$$y = 2.29 y''$$

$$x = x''$$

This is the PG estimate  
of the streamlines at  
Mach 0.9 for an NACA  
0050.

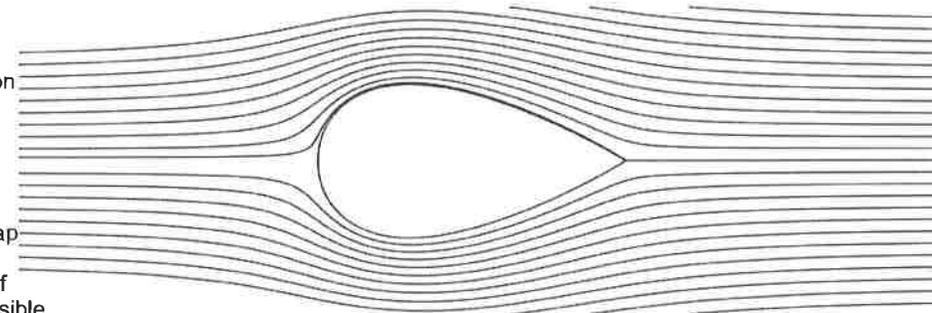
The actual solution will  
contain shocks. The PG  
transform becomes less  
accurate as M nears 1,  
since the higher order  
TSD approximation  
terms were dropped.



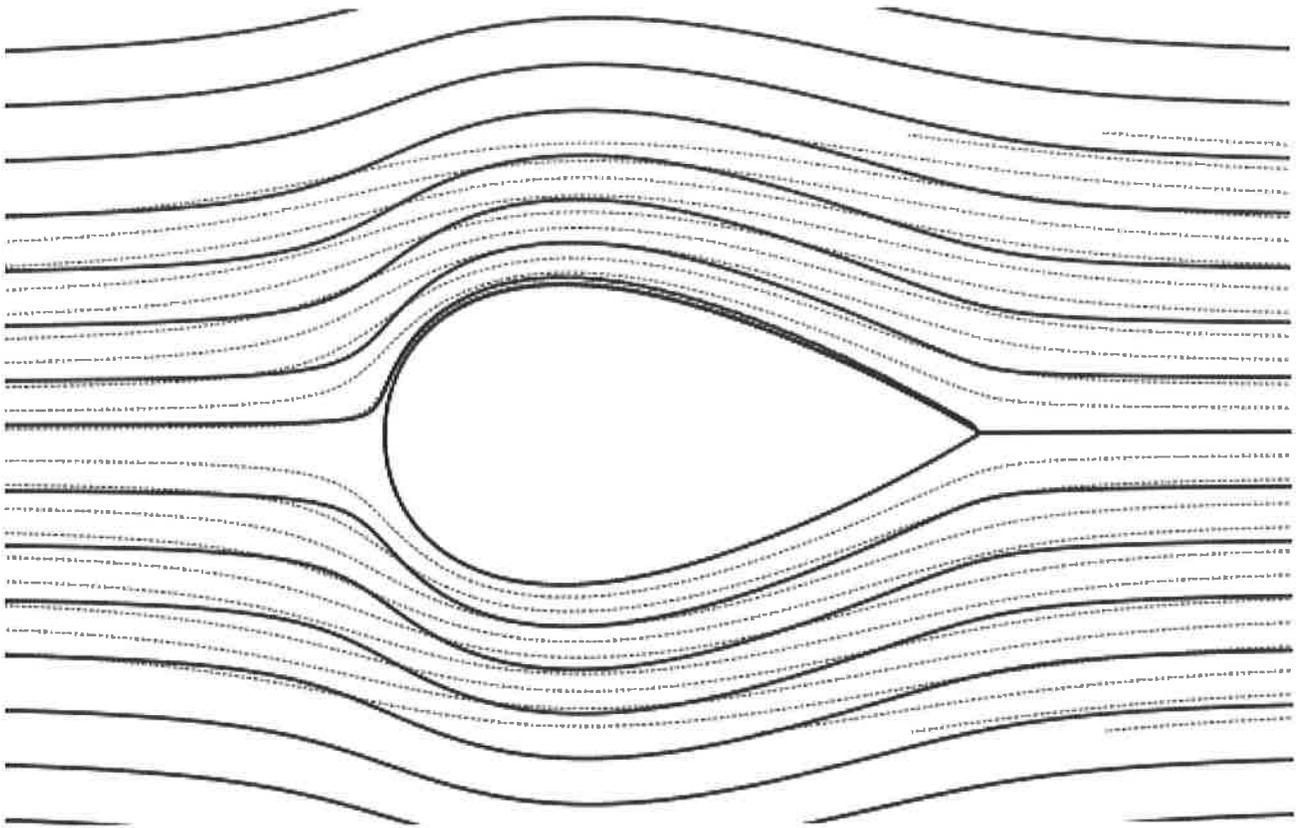
This is the equivalent  
incompressible solution  
for an NACA 0050.

Pay attention to the  
streamlines.

Remember that the gap  
between streamlines  
is a visual indication of  
velocity for incompressible  
flows.



Visual comparison of incompressible (dashed line) and compressible (solid line) for exaggerated NACA 0050 case at Mach 0.9. Consistent with previous analysis, streamlines in compressible flows tend to thicken and propagate outward further.



charles-oneill.com 2015

Recall our discussion of streamtubes.



What about pressure?

Göthert's Rule

$$C_p \equiv \frac{P - P_\infty}{\frac{1}{2} \rho_\infty V_\infty^2} = \frac{P - P_\infty}{\frac{1}{2} \gamma M_\infty^2 P_\infty} = \frac{2}{\gamma M_\infty^2} \frac{P}{P_\infty} - \frac{2}{\gamma M_\infty^2}$$

From the previous Taylor series expansion

$$\frac{P}{P_\infty} = 1 - \gamma M_\infty^2 \left( u + \frac{1}{2} (1 - M_\infty^2) u^2 + \frac{1}{2} (v^2 + w^2) \right) + \dots$$

Substitute,

$$C_p \approx \frac{2}{\gamma M_\infty^2} \left( \frac{P}{P_\infty} - 1 \right) = \frac{2}{\gamma M_\infty^2} \left( -\gamma M_\infty^2 u - \frac{1}{2} \gamma M_\infty^2 (1 - M_\infty^2) u^2 + \dots \right)$$

small product

$$= -2u$$

For a potential flow

$$C_p \approx -2\phi_x$$



$$\approx -2 \frac{d\phi}{dx} = -2 \frac{d\phi}{dx''} \frac{dx''}{dx} = -2\phi_x''$$

no change?! Wrong!

Units of  $\phi$ :

$$V_i = \frac{d}{dx_i} \phi \quad \left[ \frac{ft}{s} \right] = \left[ \frac{1}{ft} \right] [U] \Rightarrow U = \frac{ft^2}{s}$$

Oh,  $\phi$  has units...  
and must be transformed  
(see streamline spacing later)

$$\phi = \frac{1}{\beta^2} \phi'$$

$$C_p \approx \frac{-2\phi_x''}{\beta^2} = \frac{1}{\beta^2} C_p''$$

The incompressible  $C_p$  is scaled by  $\frac{1}{1-M_\infty^2}$  to give the compressible  $C_p$ .

Note that  $\gamma$  is scaled by  $\beta$  but  $C_p$  by  $\beta^2$ !

Note: This is 2D only!

$$C_L = \frac{1}{S} \int \Delta C_p dx dy = \frac{1}{S} \frac{1}{\beta^2} \int \Delta C_p'' dx'' dy'' = \frac{C_L}{\beta^2} \Rightarrow$$

Compressibility increases the slope of  $C_L$  vs  $\alpha$

2D Prandtl's Rule

$$C_{L, \text{tot } M} = C_{L, \text{tot}} \frac{1}{\sqrt{1-M_\infty^2}}$$