

Lesson 19  
Small Disturbance Supersonic Flow

# Small Disturbance Supersonic Flows

Prandtl-Glauert is valid for  $M_\infty > 1$  when slender

$$(1 - M_\infty^2) \phi_{xx} + \phi_{yy} + \phi_{zz} = 0$$

+ when  $M < 1$       ← elliptic  
 0 when  $M = 1$       ← not valid here.      parabolic  
 - when  $M > 1$       ← hyperbolic

linear PDE.

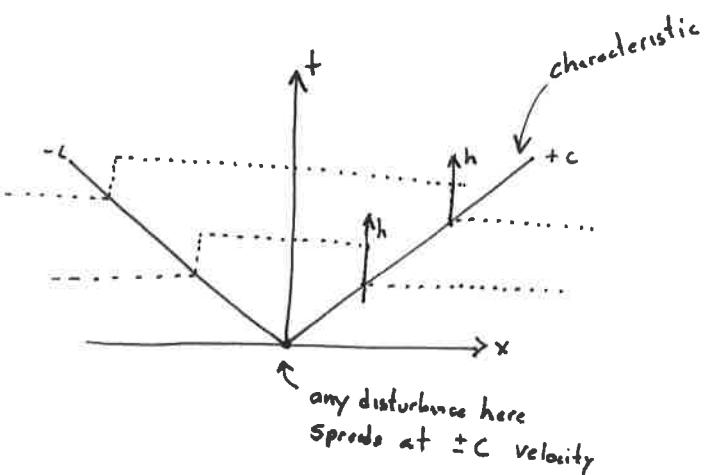
Redefine to

$$-\left(M_\infty^2 - 1\right) \phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \text{with } \beta = \sqrt{M_\infty^2 - 1}$$

$$-\beta^2 \phi_{xx} + \phi_{yy} + \phi_{zz} = 0$$

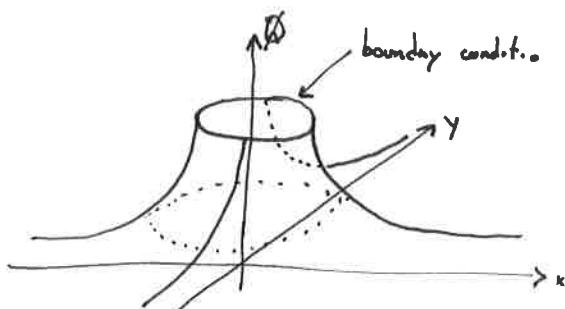
Compare to the "wave equation" from PDE

$$\frac{\partial^2 h}{\partial t^2} = c^2 \frac{\partial^2 h}{\partial x^2}$$



Compare to the Laplace equation from PDE and subsonic flow

$$\phi_{xx} + \phi_{yy} = 0$$



Disturbances die away with distance from bcs.

No characteristics.

Visually

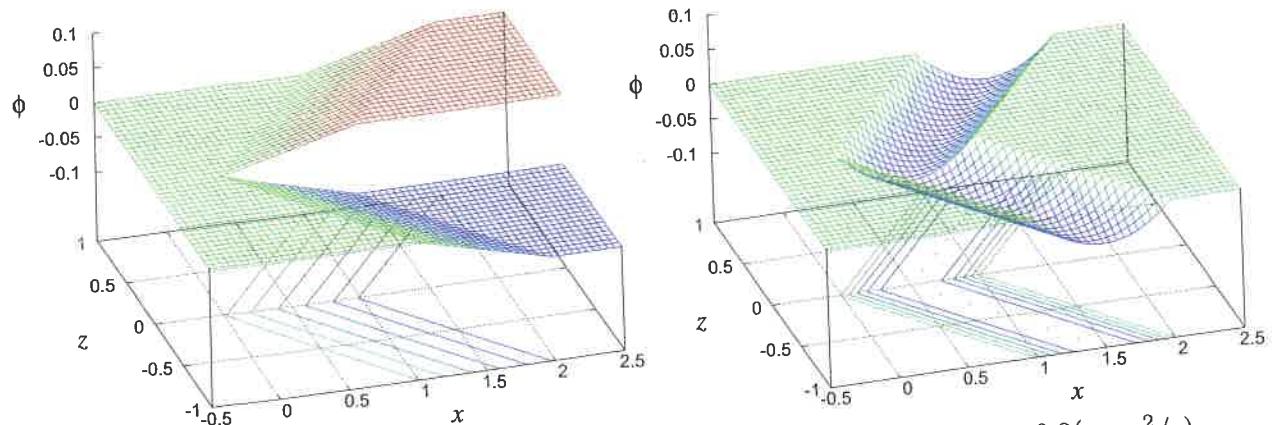
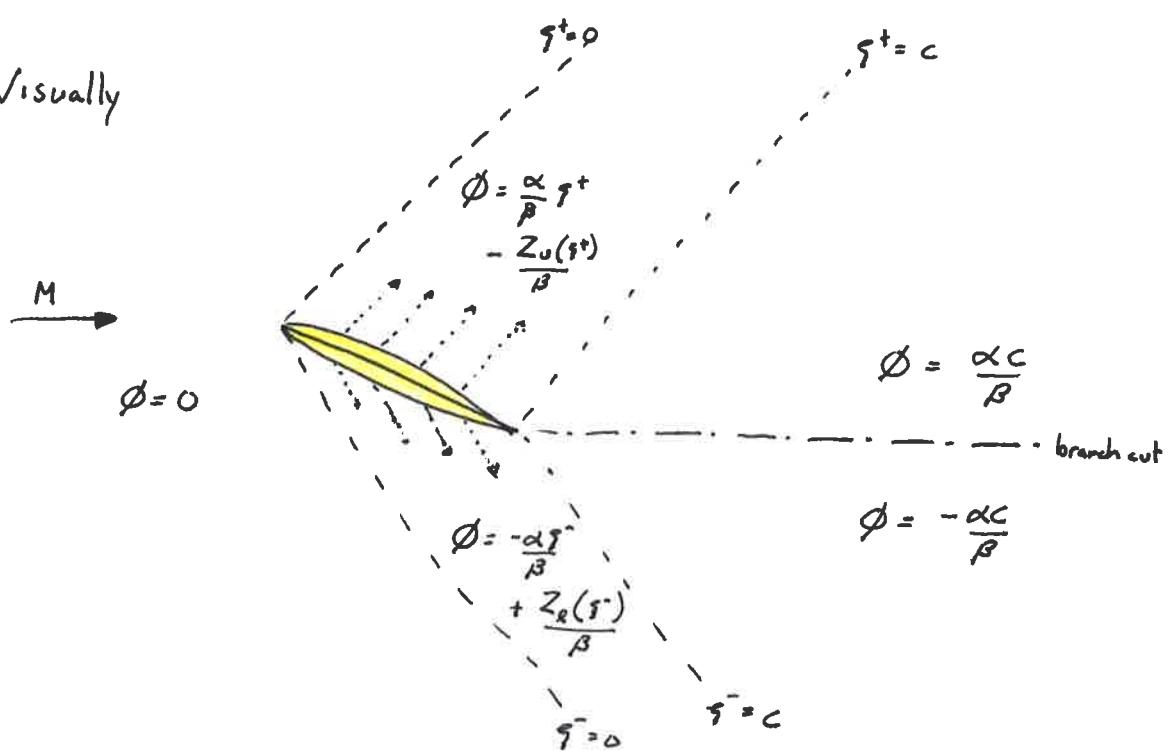
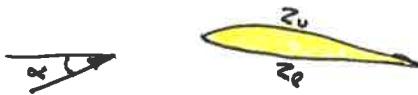


Figure 8.19: Supersonic potentials over 2D airfoil. Left:  $\alpha = 0.05$ . Right:  $Z(x) = 0.2(x - x^2/c)$ .

# Thin supersonic airfoil in 2D



$$z_e = Z_e(x)$$

Governing Equation

$$-\beta^2 \phi_{xx} + \phi_{zz} = 0$$

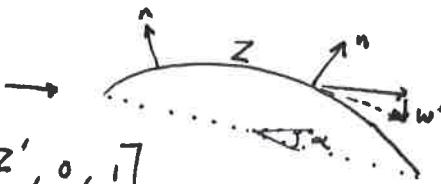
$$\beta = \sqrt{M_\infty^2 - 1}$$

Boundary Condition

No flow through surface

$$\text{Slender body so } \hat{n} = [\alpha - z', 0, 1]$$

$$\phi_z = w = -n_x = z' - \alpha$$



Solution (FVA Eq 8.132)

$$\phi = \begin{cases} 0 & \text{upstream} \\ \frac{\alpha \xi^+}{\beta} - \frac{Z_0(\xi^+)}{\beta} & \text{upper wave} \\ 0 \leq \xi^+ \leq c, z > 0 \\ -\frac{\alpha \xi^-}{\beta} + \frac{Z_e(\xi^-)}{\beta} & \text{lower wave} \\ 0 \leq \xi^- \leq c, z < 0 \\ \pm \frac{\alpha}{\beta} c & \text{downstream} \\ \xi^\pm > c \end{cases}$$

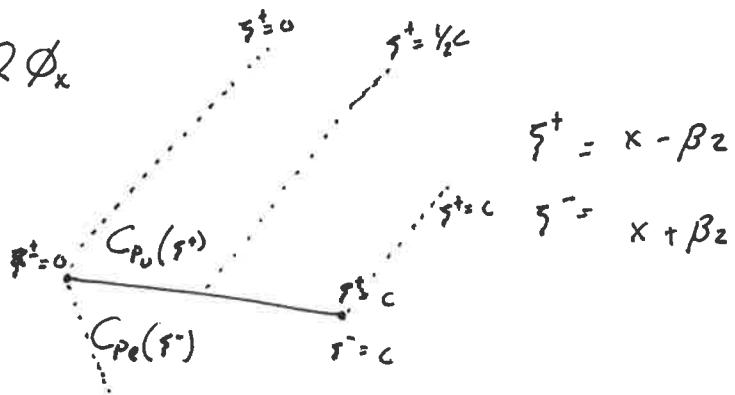
$\phi$  constant along characteristics  $\xi^+$  and  $\xi^-$

$$\xi^+ = x - \beta z$$

$$\xi^- = x + \beta z$$

Pressure

$$C_p = -2\phi_x$$



$$\begin{aligned} C_{p0} = -2\phi_x(s^+) &= -2 \frac{d\phi}{ds^+} \frac{ds^+}{dx} = -2 \frac{d}{ds^+} \left( \frac{\alpha}{\beta} s^+ - Z_0(s^+) \right) \frac{d}{dx} (x - \beta z) \\ &= -2 \left( \frac{\alpha}{\beta} - \frac{1}{\beta} \frac{dZ_0}{ds^+} \right) \end{aligned}$$

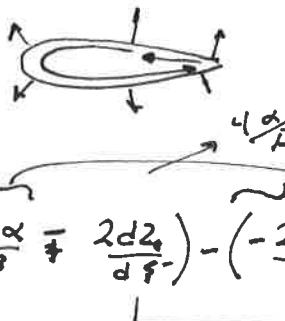
$$C_{p1} = -2\phi_x(s^-) = \dots = -2 \left( \frac{\alpha}{\beta} + \frac{dZ_0}{ds^-} \right)$$

$$\frac{dZ_0}{ds^+} = Z'_0$$

Lift

$$\begin{aligned} C_L &= \int -C_p \hat{n} \cdot \hat{z} \frac{dx}{c} \\ &= \int_0^1 C_{p1} - C_{p0} \frac{dx}{c} = \int_0^1 \left( \frac{2\alpha}{\beta} + \frac{2dZ_0}{ds^-} \right) - \left( -\frac{2\alpha}{\beta} + \frac{2}{\beta} \frac{dZ_0}{ds^+} \right) \frac{dx}{c} \\ &= \int_0^1 \frac{4\alpha}{\beta} \frac{dx}{c} \end{aligned}$$

integral of derivatives  
that are 0 at the  
limits  $\rightarrow 0$



$$C_L = \frac{4\alpha}{\beta}$$

Slender Supersonic theory gives a lift coefficient slope of  $\frac{4}{\beta}$  and no contribution from thickness or camber.

$$\begin{aligned} C_{L\alpha} &= \frac{4}{\beta} \\ &= \frac{4}{\sqrt{M_\infty^2 - 1}} \end{aligned}$$

(8) Cont

$$D' = - \int \rho \left[ \left( U_{\infty} + \phi_x U_{\infty} \right) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \left[ \left( U_{\infty} + \phi_x U_{\infty} \right) - \begin{pmatrix} U_{\infty} \\ 0 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= - \int_c^0 \rho \phi_x U_{\infty} \phi_x U_{\infty} d\xi^+$$

$$= - \int_c^0 \rho \frac{d\phi}{d\xi^+} (-\beta) \frac{d\phi}{d\xi^+} U_{\infty}^2 d\xi^+$$

$$= + \rho U_{\infty}^2 \beta \int_c^0 \left( \frac{d\phi}{d\xi^+} \right)^2 d\xi^+$$

On the upper surface,  $\phi = \frac{\alpha \xi^+}{\beta} - \frac{1}{\beta} Z_0(\xi^+)$

$$\phi_{\xi^+} = \frac{\alpha}{\beta} - \frac{Z'_0}{\beta} \quad \text{when } Z' = \frac{dz}{d\xi}$$

$$D' = \rho U_{\infty}^2 \beta \int_c^0 \left( \frac{\alpha}{\beta} - \frac{Z'_0}{\beta} \right)^2 d\xi^+$$

~~ReAssume small angles / slender body so  $\alpha Z'_0 \approx 0$~~  No!

$$D' \approx \rho U_{\infty}^2 \beta \left[ \int_c^0 \left( \frac{\alpha}{\beta} \right)^2 d\xi^+ + \int_c^0 \left( \frac{Z'_0}{\beta} \right)^2 d\xi^+ \right]$$

Mixed terms are zero

$$\int_c^0 \alpha Z'_0 d\xi = 0$$

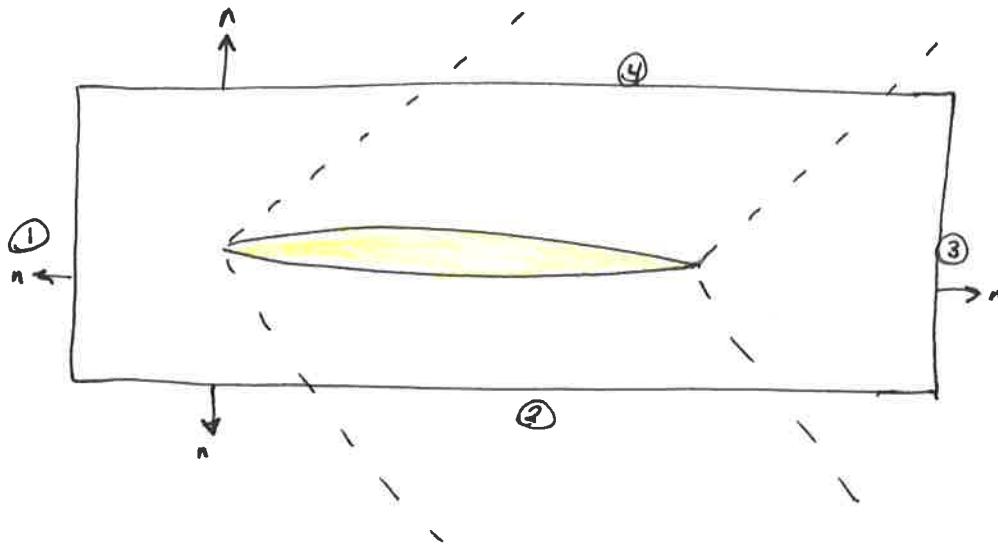
since  $Z(0) = z(c) = 0$

$$= \rho U_{\infty}^2 \beta \frac{\alpha^2}{\beta^2} c + \rho U_{\infty}^2 \beta \frac{1}{\beta^2} \int_c^0 (Z'_0)^2 d\xi^+$$

## Drag from Momentum Conservation

Warning: Since characteristics extend "forever", defining a "large" integration contour integral is not sufficient for terms to be ignored.

$$D' = - \oint \rho (\mathbf{V} \cdot \hat{\mathbf{n}}) (\mathbf{V} - \mathbf{V}_\infty) \hat{x} - (P_\infty - P) \hat{\mathbf{n}} \cdot d\mathbf{s}$$



$$\textcircled{1} \quad \phi = 0 \Rightarrow \mathbf{V} - \mathbf{V}_\infty = 0 \quad \text{and} \quad P = P_\infty \Rightarrow D'_1 = 0$$

$$\textcircled{2} \quad U' = \phi_x V_\infty \quad \text{and} \quad W' = \phi_z V_\infty \Rightarrow U = U_\infty + \phi_x V_\infty \quad W = \phi_z V_\infty$$

$$\xi^+ = x - \beta z \quad \text{such that} \quad \frac{d\phi}{dz} = \frac{d\phi}{d\xi^+} \frac{d\xi^+}{dz} = \frac{d\phi}{d\xi^+} (-\beta)$$

$$\frac{d\phi}{dx} = \frac{d\phi}{d\xi^+} \frac{d\xi^+}{dx} = \frac{d\phi}{d\xi^+}$$

In the normal direction  $(0, 1)$ , the pressure is  $C_{p_0} = \frac{2}{\rho} (-\alpha + Z'_0(\xi^+))$   
 But since this only acts in the  $\mathbf{n}$  direction, we will ignore it (only useful for  $C_a$ ).

③ No perturbation  $D' = 0$

You could add a wake model here for viscous drag.

② By symmetry and inspection.

$$\xi^- = x + \beta z \Rightarrow \frac{d\phi}{dz} = \frac{d\phi}{d\xi} - \beta$$

$$\text{and } \hat{n}_x = -\hat{n}_y$$

$$D' = \rho U_\infty^2 \frac{\alpha^2}{\beta} + \frac{\rho U_\infty^2}{\beta} \int_0^c (Z'_u)^2 dz^+$$

Total

$$C_d = \frac{D}{\frac{1}{2} \rho V_\infty^2 c} = \frac{2 \rho U_\infty^2 \frac{\alpha^2}{\beta}}{\frac{1}{2} \rho V_\infty^2 c} + \frac{\rho U_\infty^2}{\frac{1}{2} \rho V_\infty^2 c} \int_0^c (Z'_u)^2 + (Z'_v)^2 dz^+$$

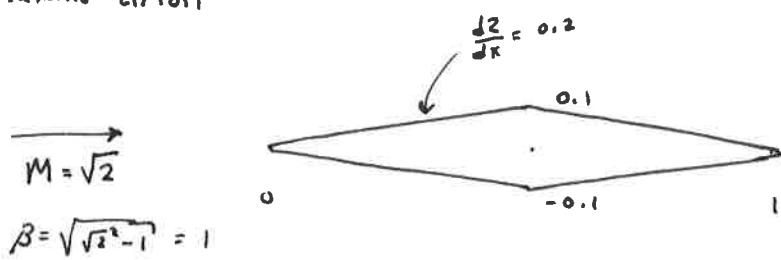
$$C_d = \underbrace{4 \frac{\alpha^2}{\beta}}_{C_d \alpha} + \frac{2}{\beta} \int_0^c (Z'_u)^2 + (Z'_v)^2 dz^+$$

Wave drag depends on  $\alpha^2$  and the square of surface slope.

or

Wave drag depends on lift and shape

Ex: Diamond airfoil



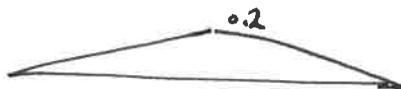
$$C_e = \frac{4}{1} \alpha$$

$$C_d = 4\alpha^2 + 2 \left[ \int_0^{0.5} (0.2)^2 + (-0.2)^2 ds \right] \cdot 2 = 4\alpha^2 + \cancel{0.16} \cancel{cosine} 0.16$$

$$\frac{C_e}{C_d} = \frac{4\alpha}{4\alpha^2 + \cancel{0.16}} \quad \text{or generally} \quad \frac{C_e}{C_d} = \frac{4\alpha}{4\alpha^2 + 16(\frac{t}{c})^2}$$

Max  $\frac{C_e}{C_d}$  is at  $\alpha = 2 \frac{t}{c}$

Ex: Cambered diamond



$$C_e = 4\alpha$$

$$C_d = 4\alpha^2 + 2 \cdot \left[ \int_0^{0.5} (0.4)^2 ds + \int_{0.5}^1 (-0.4)^2 ds \right]$$

$$= 4\alpha^2 + 2 \cdot \left[ 0.5 \cdot 0.64 + 0.5 \cdot (0.48) \right] = 4\alpha^2 + \cancel{0.32}$$

Same thickness, same lift vs  $\alpha$ , double the drag.

Minimum  $C_d$

$$\text{Drag} = \frac{2}{\beta} \int_0^l 2(y')^2 dx = \frac{4}{\beta} \int_0^l (y')^2 dx$$

What shape minimizes  $C_d$ ?

Euler-Lagrange "Calculus of Variations"

$$J = \int F dx$$

$$\frac{dF}{dy} - \frac{d}{dx} \left( \frac{dF}{dy'} \right) = 0 \quad \leftarrow \text{Convert integral into differential equation!}$$

$$F = (y')^2$$

$$\frac{dF}{dy} = 0 \quad \frac{dF}{dy'} = 2y'$$

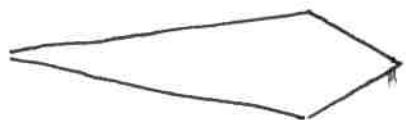
$$0 - \frac{d}{dx}(2y') = -2y'' = 0$$

$$\text{with } y(0)=0 \\ y(l)=0$$

Find a curve with  $y''=0$  such that  $y(0)=y(l)=0$

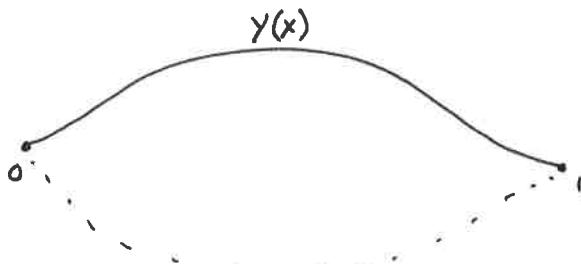


Slender such that the perturbations are scaled with  $C_D$  and length.



(i.e., this shape also works)

Ex:



$$\text{Area} = 2 \int_0^1 y(x) dx$$

$$\text{Drag} = \frac{2}{\beta} \int_0^1 2(y')^2 dx = \frac{4}{\beta} \int_0^1 (y')^2 dx$$

What shape maximizes  $\text{Area}/\text{Drag}$ ?

Euler Lagrange "Calculus of Variations"

$$J(y) = \frac{2\beta}{4} \int_0^1 \frac{y(x)}{(y')^2} dx \Rightarrow F = \frac{y(x)}{(y')^2}$$

$$\underbrace{\frac{dF}{dy}}_{=0} - \underbrace{\frac{d}{dx} \left( \frac{dF}{dy'} \right)}_{=0} = 0$$

$$\frac{1}{y'^2} - \underbrace{\frac{d}{dx} \left( \frac{d y(x)}{dy' y'^2} \right)}_{=0} = 0$$

$$\frac{1}{y'^2} - \left( y'' - \frac{2y y''}{(y')^3} \right) = 0$$

mult by  $y'^3$

$$y' - y'^2 + 2yy'' = 0$$

with

$$y(0) = y(1) = 0$$

# Pitch Moment



$$dm_{x_0} = (P_u - P_e)(x - x_0) dx$$

Nondimensionalize

$$dC_{m_{x_0}} = (C_{P_u} - C_{P_e}) \left( \frac{x - x_0}{c} \right) \frac{dx}{c}$$

$$C_m \equiv \frac{M}{8c^2}$$

Remember that  $C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}$

Thus,

$$dC_{m_{x_0}} = \underbrace{\frac{2(\theta_u - \theta_e)}{\sqrt{M_\infty^2 - 1}}}_{\beta} \left( \frac{x - x_0}{c} \right) \frac{dx}{c}$$

Integrate

$$\int_0^c dC_{m_{x_0}} = \int_0^c \frac{2}{\beta} \underbrace{\left( \frac{dz_u}{dx} - \alpha + \frac{dz_e}{dx} - \alpha \right)}_{-2\alpha + 2 \text{ average } (z'_u \text{ and } z'_e)} \left( \frac{x - x_0}{c} \right) \frac{dx}{c}$$

$$-2\alpha + 2 \text{ average } (z'_u \text{ and } z'_e) = -2\alpha + 2 Z'_c$$

$$\text{So, } \theta_e \approx -\frac{dz_e}{dx} + \alpha$$

$$\text{and } \theta_u = \frac{dz_u}{dx} - \alpha$$

$$C_{m_{x_0}} = \underbrace{\int_0^c \frac{2}{\beta} (-2\alpha) \frac{x - x_0}{c} \frac{dx}{c}}_{-4\alpha \left( \frac{x^2}{2c} - \frac{x_0 x}{c} \right) \Big|_0^c} + \underbrace{\int_0^c \frac{2}{\beta} 2 Z'_c \frac{x - x_0}{c} \frac{dx}{c}}_{\frac{4}{\beta} \int_0^c Z'_c \frac{x - x_0}{c} \frac{1}{c} dx}$$

$$-4\alpha \left( \frac{c^2}{2c^2} - \frac{x_0 c}{c^2} \right)$$

$$C_{m_{x_0}} = -\underbrace{\frac{4\alpha}{\sqrt{M_\infty^2 - 1}}}_{+} + \underbrace{\frac{4}{\sqrt{M_\infty^2 - 1}} \int_0^c Z'_c \frac{x - x_0}{c} \frac{1}{c} dx}_{-}$$

Aerodynamic Center at  $x_0$  where  $\frac{dC_m}{dx} = 0$

Thus  $X_{ac} = \frac{1}{2} c$

to make  $\left( \frac{1}{2} - \frac{x_0}{c} \right) = 0$

• Mid chord